Complex Analysis, Spring 2011.

Instructor: Dmitry Ryabogin

Assignment XI.

1. **Problem 1.**

   a) Compute
   $$\int_{|z|=r} x dz$$
   by use of a parameter.

   b) Compute the integral in a) by observing that $x = (z+\bar{z})/2 = (z + r^2/z)/2$ on the circle.

   c) Compute
   $$\int_{|z|=2} \frac{dz}{z^2-1}$$
   for the positive sense of the circle.

   d) Compute
   $$\int_{|z|=1} |z-1||dz|$$

2. **Problem 2.**

   a) Suppose that $f(z)$ is analytic on a closed curve γ (i. e., f is analytic in a region that contains γ). Show that
   $$\int_{\gamma} \overline{f(z)} f'(z) dz$$
   is purely imaginary. (The continuity of $f'(z)$ is taken for granted).

   b) Assume that $f(z)$ is analytic and satisfies the inequality $|f(z)-1| < 1$ in a region Ω. Show that
   $$\int_{\gamma} \frac{f'(z)}{f(z)} dz = 0$$
   for every closed curve in Ω. (The continuity of $f'(z)$ is taken for granted).

   c) If $P(z)$ is a polynomial and C denotes the circle $|z-a| = R$, what is the value of $\int_C P(z) \overline{dz}$?

   Hint: $-2\pi i R^2 P'(a)$.

3. **Problem 3.**

a) Let $f(z)$ be continuous in the domain $\Omega = \{z \in \mathbf{C} : |z| \geq R_0, \Im z \geq a\}$, ($a$ is a fixed real number), and in this domain $f(z) \to 0$ uniformly on arcs $\{z \in \Omega, |z| = R\}$, as $z \to \infty$. Prove that for any $m > 0$,

$$\lim_{R\to\infty} \int\limits_{\Gamma_R} e^{imz} f(z) dz = 0,$$

where Γ_R is an arc of the circle $|z| = R$ in $z \in \Omega$.

Hint: On the half-arc $|z| = R$, $\Im z > 0$, use $\sin\theta \geq 2\theta/\pi$, $0 \leq \theta \leq \pi/2$. In the case $a < 0$ use the fact that the length of the corresponding arcs tends to $|a|$ as $R \to \infty$.

b) Let $f(z)$ be continuous in the half-plane $\Re z \geq \sigma$, (σ is a fixed real number), and in this half-plane $f(z) \to 0$ uniformly on arcs $\{\Re z \geq \sigma, |z| = R\}$, as $z \to \infty$. Prove that for any $t < 0$,

$$\lim_{R\to\infty} \int\limits_{\Gamma_R} e^{zt} f(z) dz = 0,$$

where Γ_R is an arc of the circle $|z| = R$, $\Re z \geq \sigma$. If $f(z)$ is continuous in $\Re z \leq \sigma$, then the statement is true, provided $t > 0$, and Γ_R is an arc of the circle $|z| = R$, $\Re z \leq \sigma$.