Complex Analysis, Spring 2011.

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Assignment XII.

1. **Problem 1.** Compute

$$a) \int\limits_{|z-\aleph|=\aleph} \frac{zdz}{z^4-1}, \ \aleph > 1; \qquad b) \int\limits_{|z|=\rho} \frac{|dz|}{|z-a|^2}, \qquad |a| \neq \rho,$$

Hint: make use of the equations $z\bar{z} = \rho^2$ and $|dz| = -i\rho\frac{dz}{z}$;

$$c) \int\limits_{|z|=5} \frac{ze^z dz}{(z-i)^3}; \qquad d) \int\limits_{|z|=1/2} z^n(1-z)^m dz; \qquad n, m \in \mathbf{Z}.$$

2. **Problem 2.**

a) Prove that a function which is analytic in the whole plane and satisfies an inequality $|f(z)| < |z|^n$ for some n and all sufficiently large $|z|$ reduces to a polynomial.

b) If $f(z)$ is analytic and $|f(z)| \le M$ for $|z| \le R$, find an upper bound for $|f^{(n)}(z)|$ in $|z| \le \rho < R$.

c) If $f(z)$ is analytic for $|z| < 1$ and $|f(z)| \le 1/(1-|z|)$, find the best estimate of $|f^{(n)}(0)|$ that Cauchy's inequality will yield.

d) Show that the successive derivatives of an analytic function at a point can never satisfy $|f^{(n)}(z)| > n!n^n$.

3. **Problem 3.**

a) According to Liouville's Theorem, a function that is analytic and bounded in the whole plane must reduce to a constant. Give another proof of this Theorem by computing

$$\int\limits_{|z|=R} \frac{f(z)dz}{(z-a)(z-b)}, \qquad |a| < R, \ |b| < R,$$

and estimating it for $R \to \infty$.

b) Let $f(z)$ be analytic in a large disc containing a closed piecewise differentiable curve γ, and let z_1, z_2,...,z_n, $z_j \neq z_k$ for $j \neq k$, be points inside γ. Prove that the integral

$$P(z) = \frac{1}{2\pi i} \int\limits_{\gamma} \frac{f(\xi)}{w_n(\xi)} \frac{w_n(\xi) - w_n(z)}{\xi - z} d\xi, \qquad w_n(z) = (z-z_1)(z-z_2)...(z-z_n),$$

is a polynomial of degree $n-1$, and $P(z_j) = f(z_j) \ \forall j = 1, ..., n$.

Hint: $(w_n(\xi) - w_n(z))/(\xi - z)$ is a polynomial in z.

4. **Problem 4.** Let γ be the piecewise differentiable closed curve that does not pass through the point a. Prove that

$$\frac{1}{2\pi i}\int_{\gamma}\frac{dz}{z-a}\in \mathbf{Z}$$

by completing the following steps.

a) Observe **formally** that

$$\int_{\gamma}\frac{dz}{z-a}=\int_{\gamma}d\log(z-a)=\int_{\gamma}d\log|z-a|+i\int_{\gamma}d\arg(z-a).$$

b) Divide γ into a finite number of subarcs such that there exists a single-valued branch of $\arg(z-a)$ on each subarc.

Hint: There exists a small circle around a that does not intersect γ. Use the compactness argument to divide γ into small subarcs with lengths smaller than that one of the circle.