1. Problem 1.
   a) Write Taylor’s formula with \( n = 3 \) for
      \[ f(z) = e^{z \sin z}; \quad \square f(z) = (1 + z)^z; \quad \exists f(z) = e^{e^z}. \]
   b) Find the order of all zeros of
      \[ f(z) = z^2 + 9; \quad \square f(z) = z \sin z; \quad \exists f(z) = (1 - e^z)(z^2 - 4)^3. \]
   c) Does there exist a function, analytic in \( z = 0 \) and for \( z = 1/k, k = 1, 2, ..., \) taking the values
      \[ \square \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{5}{6}, \frac{6}{7}, \ldots, \frac{k}{k+1}, \ldots; \quad \exists \frac{1}{2}, \frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \ldots, \frac{1}{2k}, \frac{1}{2k}, \ldots? \]

2. Problem 2.
   a) If \( f(z) \) and \( g(z) \) have the algebraic orders \( h_f \) and \( h_g \) at \( z = a \), show that \( \square f g \) has the order \( h_f + h_g \), \( \square f/g \) the order \( h_f - h_g \), and \( \exists f + g \) an order which does not exceed \( \max(h_f, h_g) \).
   b) Show that a function which is analytic in the whole plane and has a nonessential singularity at \( \infty \) reduces to a polynomial.
   c) Show that the functions \( e^z, \sin z, \) and \( \cos z \) have essential singularities at \( \infty \).

3. Problem 3.
   a) Show that any function which is meromorphic in the extended plane is rational.
   b) Prove that an isolated singularity of \( f(z) \) is removable as soon as either \( \Re f(z) \) or \( \Im f(z) \) is bounded above or below.
   \textbf{Hint}: Apply a fractional linear transformation.
   c) Show that an isolated singularity of \( f(z) \) cannot be a pole of \( e^{f(z)} \).
   \textbf{Hint}: \( f \) and \( e^{f(z)} \) cannot have a common pole (why?). Now apply the Theorem about essential singularities.

   a) Prove that
      \[ I = \int_{0}^{\infty} e^{-\rho^2} d\rho = \frac{\sqrt{\pi}}{2}. \]
**Hint:** Observe that
\[ 4I^2 = \int_{\mathbb{R}^2} e^{-r^2} d\rho d\tau \]
and use the polar coordinates.

b) Evaluate Fresnel’s integrals
\[ \int_0^\infty \cos x^2 dx = \int_0^\infty \sin x^2 dx = \frac{\sqrt{\pi}}{2\sqrt{2}}. \]

**Hint:** Integrate \( f(z) = e^{iz^2} \) over the boundary of
\[ \Omega = \{ z \in \mathbb{C} : 0 \leq |z| \leq R, \, 0 \leq \arg z \leq \pi/4 \} \]
and use the reasons similar to those related to Problem 3 a) from Assignment XI.