Complex Analysis, Spring 2011.

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Assignment XIII.

1. **Problem 1.**

   a) Write Taylor's formula with $n = 3$ for

   $$\aleph)\ f(z) = e^{z\sin z}; \qquad \beth)\ f(z) = (1+z)^z, \qquad \gimel)\ f(z) = e^{e^z}.$$

   b) Find the order of all zeros of

   $$\aleph)\ f(z) = z^2 + 9; \qquad \beth)\ f(z) = z\sin z, \qquad \gimel)\ f(z) = (1-e^z)(z^2-4)^3.$$

   c) Does there exist a function, analytic in $z = 0$ and for $z = 1/k$, $k = 1, 2, ...$, taking the values

   $$\aleph)\ \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{5}{6}, \frac{6}{7}, ..., \frac{k}{k+1}, ...; \qquad \beth)\ \frac{1}{2}, \frac{1}{2}, \frac{1}{4}, \frac{1}{4}, \frac{1}{6}, \frac{1}{6}, ..., \frac{1}{2k}, \frac{1}{2k}, ...?$$

2. **Problem 2.**

   a) If $f(z)$ and $g(z)$ have the algebraic orders h_f and h_g at $z = a$, show that $\aleph)$ fg has the order $h_f + h_g$, $\beth)$ f/g the order $h_f - h_g$, and $\gimel)$ $f + g$ an order which does not exceed $\max(h_f, h_g)$.

   b) Show that a function which is analytic in the whole plane and has a nonessential singularity at ∞ reduces to a polynomial.

   c) Show that the functions e^z, $\sin z$, and $\cos z$ have essential singularities at ∞.

3. **Problem 3.**

   a) Show that any function which is meromorphic in the extended plane is rational.

   b) Prove that an isolated singularity of $f(z)$ is removable as soon as either $\Re f(z)$ or $\Im f(z)$ is bounded above or below.

   Hint: Apply a fractional linear transformation.

   c) Show that an isolated singularity of $f(z)$ cannot be a pole of $e^{f(z)}$.

   Hint: f and e^f cannot have a common pole (why?). Now apply the Theorem about essential singularities.

4. **Problem 4.**

   a) Prove that

   $$I = \int_0^\infty e^{-\rho^2} d\rho = \frac{\sqrt{\pi}}{2}.$$

Hint: Observe that

$$4I^2 = \int_{\mathbf{R^2}} e^{-\rho^2 - r^2} d\rho dr$$

and use the polar coordinates.

b) Evaluate Fresnel's integrals

$$\int_0^\infty \cos x^2 dx = \int_0^\infty \sin x^2 dx = \frac{\sqrt{\pi}}{2\sqrt{2}}.$$

Hint: Integrate $f(z) = e^{iz^2}$ over the boundary of

$$\Omega = \{z \in \mathbf{C} : \ 0 \le |z| \le R, \ 0 \le \arg z \le \pi/4\}$$

and use the reasons similar to those related to Problem 3 a) from Assignment XI.