Complex Analysis, Spring 2011.

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Assignment XIV.

1. **Problem 1.**

   a) Find isolated singularities and describe the behavior of the function near ∞:

   $$\aleph) \ \frac{1}{e^z-1}-\frac{1}{z}; \qquad \beth) \ e^{z/(1-z)}; \qquad \gimel) \ \sin\left(\frac{1}{\sin(1/z)}\right).$$

   b) Construct examples of functions having in the extended plane only the following singularities: the pole of the order m at 0 and the pole of the order n at ∞;

2. **Problem 2.**

   a) Determine explicitly the largest disk about the origin whose image under the mapping $w = z^2 + z$ is one to one.

   b) Same problem for $w = e^z$.

   c) Apply the representation $f(z) = w_0 + \xi(z)^n$ to $\cos z$ with $z_0 = 0$. Determine $\xi(z)$ explicitly.

   d) If $f(z)$ is analytic at the origin and $f'(0) \neq 0$, prove the existence of an analytic $g(z)$ such that $f(z^n) = f(0) + g(z)^n$ in a neighborhood of 0.

3. **Problem 3.**

   a) Show that $|f(z)| \leq 1$ for $|z| \leq 1$ implies

   $$\frac{|f'(z)|}{1-|f(z)|^2} \leq \frac{1}{1-|z|^2}.$$

   b) If $f(z)$ is analytic and $\Im f(z) \geq 0$ for $\Im z > 0$, show that

   $$\frac{|f(z)-f(z_0)|}{|f(z)-\overline{f(z_0)}|} \leq \frac{|z-z_0|}{|z-\bar{z}_0|}, \qquad \frac{|f'(z)|}{\Im f(z)} \leq \frac{1}{y}.$$

   c) In a) and b) prove that equality implies that $f(z)$ is a fractional linear transformation.

   d) Derive the corresponding inequalities if $f(z)$ maps $|z| < 1$ into the upper half plane.

   e) Prove by use of Schwarz's lemma that every one-to-one conformal mapping of a disk onto another (or a half plane) is given by a fractional linear transformation.

4. **Problem 4.**

a) Let f be analytic in a square centered at the origin with sides, say, h_1, h_2, h_3, h_4. Assume that $|f(z)| \leq 1\ \forall z \in h_j,\ j = 1, 2, 3$, and $|f(z)| \leq 16\ \forall z \in h_4$. Give the best possible (in your opinion) upper bound for $|f(0)|$.

b) Let A be a set of functions f analytic in the strip $|\Im z| \leq \pi$, and such that $|f(\pm i\pi)| \leq 1$. Is it possible to give an upper bound for $\sup\limits_{f \in A} |f(0)|$?

Hint: Consider e^{e^z}.