# Complex Analysis, Spring 2011.

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# Assignment II.

1. **Problem 1.**

   a) Describe curves defined by the following equations:

   א) $z = t + i/t$; ב) $z = t^2 + it^4$, $-\infty < t < \infty$; ג) $z = -t + i\sqrt{1-t^2}$, $-1 \le t \le 1$.

   b) For a map $w = z^2$ find the images of the lines $x = C$, $y = C$, $x = y$, $|z| = R$, $arg z = \alpha$. Are these curves transformed one-to-one?

   c) Find the pre-images (on $z$-plane) of $u = C$, $v = C$, $(w = z^2 = u + iv)$.

   d) Find the domain where $f(z) = |x^2 - y^2| + 2i|xy|$ is analytic.

2. **Problem 2.**

   a) If $g(w)$ and $f(z)$ are analytic functions, show that $g(f(z))$ is also analytic.

   b) Show that an analytic function cannot have a constant absolute value without reducing to a constant.

   c) Prove rigorously that $f(z)$ and $\overline{f(\bar{z})}$ are simultaneously analytic.

3. **Problem 3.**

   a) Find the most general harmonic polynomial of the form $ax^3 + bx^2y + cxy^2 + dy^3$. Determine the conjugate harmonic function and the corresponding analytic function by integration and by the formal method.

   b) Prove that the functions $u(z)$ and $u(\bar{z})$ are simultaneously harmonic.

   c) Show that a harmonic function satisfies the formal differential equation

   $$\frac{\partial^2 u}{\partial z \partial \bar{z}} = 0.$$

   d) Are $|f(z)|$, $arg f(z)$, $\ln|f(z)|$ harmonic, provided $f(z)$ is analytic?

   e) Let $u$ be harmonic. Is $u^2$ harmonic?

4. **Problem 4.** Following the steps outlined below prove the *Lucas's theorem: the smallest convex polygon that contains the zeros of a polynomial $P(z)$ also contains the zeros of the derivative $P'(z)$.*

   a) Observe that it is enough to prove that if all zeros of $P(z)$ lie in a half plane, then all zeros of $P'(z)$ lie in the same half plane.

   b) To prove a) show at first that

   $$\frac{P'(z)}{P(z)} = \sum_{j=1}^{n} \frac{1}{z - \alpha_j}.$$

c) Assume that all $\alpha_k$ are in the half plane $H$ defined as $Im(z-a)/b < 0$ but a root $z$ of $P'(z)$ is not there. To get a contradiction look at $Im(bP'(z)/P(z))$. What can you say about $Im(z-\alpha_k)/b$?