Complex Analysis, Spring 2011.

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Assignment III.

1. **Problem 1.**

   a) Prove that

   $$\frac{\partial}{\partial \bar{z}}(f\,g) = \frac{\partial f}{\partial \bar{z}} g + \frac{\partial g}{\partial \bar{z}} f, \qquad \frac{\partial}{\partial z}(f\,g) = \frac{\partial f}{\partial z} g + \frac{\partial g}{\partial z} f.$$

   b) Let $f(z)$ and $zf(z)$ be two **complex**-valued harmonic functions (their real and imaginary parts both satisfy the Laplace equation). Prove that f is analytic.

2. **Problem 2.**

   a) Is $e^{-1/|z|}$ uniformly continuous in $0 < |z| \leq 1$?

   b) Let $f(z)$ be uniformly continuous in $|z| < 1$. Prove that for any point ξ on the circle $|z| = 1$, and for any sequence $z_n \to \xi$, $|z_n| < 1$, there exists a limit $\lim_{n\to\infty} f(z_n)$.

   c) Prove also that the limit does not depend on the choice of the sequence $\{z_n\}$, and that the function f, defined on the boundary as $f(\xi) = \lim_{n\to\infty} f(z_n)$, is continuous in the **closed** disc $|z| \leq 1$.

3. **Problem 3.**

   a) Use the method described in class to develop

   $$\frac{z^4}{z^3 - 1}$$

   into partial fractions.

   b) If $R(z)$ is a rational function of order n, how large and how small can the order of $R'(z)$ be?

4. **Problem 4.**

   a) If Q is a polynomial with distinct roots $\alpha_1, ..., \alpha_n$, and if P is a polynomial of degree $< n$, show that

   $$\frac{P(z)}{Q(z)} = \sum_{k=1}^{n} \frac{P(\alpha_k)}{Q'(\alpha_k)(z - \alpha_k)}.$$

   b) Use the above formula to prove that there exists a unique polynomial P of degree $< n$ with given values c_k at the points α_k (Lagrange's interpolation polynomial).

5. **Problem 5.** If a rational function is real on $|z| = 1$, how are the zeros and the poles situated?