Complex Analysis, Spring 2011.
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Assignment IV.

1. Problem 1.
   a) Expand \((1 - z)^{-m}\) \(m\) a positive integer, in powers of \(z\).
   b) Expand \(\frac{2z+3}{z+1}\) in powers of \(z - 1\). What is the radius of convergence?

2. Problem 2.
   a) If \(\sum a_n z^n\) has radius of convergence \(R\), what is the radius of convergence of \(\sum a_n z^{2n}\)?
   b) If \(f(z) = \sum a_n z^n\), what is \(\sum n^3 a_n z^n\)?
   c) If \(\sum a_n z^n\) and \(\sum b_n z^n\) have radii of convergence \(R_1\) and \(R_2\), show that the radius of convergence of \(\sum a_n b_n z^n\) is at least \(R_1, R_2\).
   d) If \(\lim_{n \to \infty} |a_{n+1}|/|a_n| = R\), prove that \(\sum a_n z^n\) has radius of convergence \(R\).

   Hint: Consider
   \[
   |a_n| = \left| \frac{a_n}{a_{n-1}} \right| \left| \frac{a_{n-1}}{a_{n-2}} \right| \ldots \left| \frac{a_{k+1}}{a_k} \right| |a_k|
   \]
   to show that
   \[
   \lim_{n \to \infty} |a_{n+1}|/|a_n| = R \quad \text{yields} \quad \lim_{n \to \infty} |a_n|^{1/n} = R.
   \]

3. Problem 3.
   a) Find the radius of convergence of the following power series:
   \(\Re) \sum n^p z^n\), \(\Im) \sum \frac{z^n}{n!}\), \(\Im) \sum q^{n^2} z^n (|q| < 1)\), \(\Im) \sum z^n!\).
   b) Describe the convergence of the series in a) for \(|z| = 1\).

   a) For what values of \(z\) is
   \[
   \sum_{n=0}^{\infty} \left( \frac{z}{1 + z} \right)^n
   \]
   convergent?
   b) Same question for
   \[
   \sum_{n=0}^{\infty} \frac{z^n}{1 + z^{2n}}.
   \]
   \text{Hint: Does there exists any difference between } |z| < 1 \text{ and } |z| > 1?
5. Problem 5*. The purpose of this exercise is to prove a weak version of the following Theorem of Levy and Steinitz: If a series \( A = z_1 + z_2 + z_3 + \ldots \) converges, but not absolutely, then there exists a line \( l \) such that for every \( s \in l \) there exists a permutation \( \sigma(n) \) of indices of \( A \), such that the permuted series \( A' = a_{\sigma(1)} + a_{\sigma(2)} + a_{\sigma(3)} + \ldots \) converges to \( s \).

Definition. Let \( \{r_n\}, \{s_m\} \) be two increasing sequences of natural numbers, such that \( r_n < r_{n+1}, s_m < s_{m+1}, r_m \neq s_m \ \forall m, n \), and \( \cup_{n,m}(r_n \cup s_m) = \mathbb{N} \). Then two series
\[
A_r := z_{r_1} + z_{r_2} + z_{r_3} + \ldots \quad \text{and} \quad A_s := z_{s_1} + z_{s_2} + z_{s_3} + \ldots
\]
are called complemented parts of a series \( A = z_1 + z_2 + z_3 + \ldots \) (In other words, you color the terms of \( A \) in red and blue, say).

If a permutation \( \sigma, \sigma : \mathbb{N} \to \mathbb{N} \), of terms of \( A \) does not change the order inside \( A_r, A_s \) (if \( z_{r_m} \) appears before \( z_{r_n} \) and \( z_{s_m} \) appears before \( z_{s_n} \) for any \( m, n, m < n \), after permutation), then one says that \( \sigma \) moves complemented parts \( A_r, A_s \) with respect to each other.

Prove a)- f) and obtain the proof.

a) The series \( A_s(A_r) \) is convergent, provided \( A \) and \( A_r(A_s) \) are convergent. If a permutation \( \sigma \) moves complemented parts \( A_r, A_s \) with respect to each other, then the sum \( z_{\sigma(1)} + z_{\sigma(2)} + z_{\sigma(3)} + \ldots \) remains unchanged (is \( A \)).

b) If a real \( A \) is convergent, but not absolutely, and if one of the complemented parts, say, \( A_r \), diverges to \( \infty \), then another complemented part, \( A_s \) diverges to \( -\infty \).

c) If a real \( A \) is convergent, but not absolutely, and if all the terms of one of the complemented parts, say, \( A_r \), have the same signs, then, using only the permutations that move the complemented parts, one can obtain any sum for the permuted series.

d) If a series \( |z_1| + |z_2| + |z_3| + \ldots \) diverges, then there exists a direction of concentration \( \alpha \), such that \( \forall \epsilon > 0 \), the series of absolute values of terms that are located in the angle \( \alpha - \epsilon < \arg z < \alpha + \epsilon \) is divergent.

e) If \( A \) is convergent, but not absolutely, and if a direction of concentration of \( A \) is a positive part of the real axes, prove that one can find \( \{r_n\} \) such that \( \text{Re}z_{r_1} + \text{Re}z_{r_2} + \text{Re}z_{r_3} + \ldots \to \infty \), but \( \text{Im}z_{r_1} + \text{Im}z_{r_2} + \text{Im}z_{r_3} + \ldots \) is convergent.

f) Prove the Theorem of Levy and Steinitz.