Complex Analysis, Spring 2011.

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Assignment IV.

1. **Problem 1.**

   a) Expand $(1-z)^{-m}$ m a positive integer, in powers of z.

   b) Expand $\frac{2z+3}{z+1}$ in powers of $z-1$. What is the radius of convergence?

2. **Problem 2.**

   a) If $\sum a_n z^n$ has radius of convergence R, what is the radius of convergence of $\sum a_n z^{2n}$? of $\sum a_n^2 z^n$?

   b) If $f(z) = \sum a_n z^n$, what is $\sum n^3 a_n z^n$?

   c) If $\sum a_n z^n$ and $\sum b_n z^n$ have radii of convergence R_1 and R_2, show that the radius of convergence of $\sum a_n b_n z^n$ is at least R_1, R_2.

   d) If $\lim\limits_{n\to\infty} |a_{n+1}|/|a_n| = R$, prove that $\sum a_n z^n$ has radius of convergence R.

   Hint: Consider

   $$|a_n| = \frac{|a_n|}{|a_{n-1}|}\frac{|a_{n-1}|}{|a_{n-2}|} \cdots \frac{|a_{k+1}|}{|a_k|}|a_k|$$

   to show that

   $$\lim_{n\to\infty} |a_{n+1}|/|a_n| = R \qquad \text{yields} \qquad \lim_{n\to\infty} |a_n|^{1/n} = R.$$

3. **Problem 3.**

   a) Find the radius of convergence of the following power series:

   $$\text{א)} \sum n^p z^n, \qquad \text{ב)} \sum \frac{z^n}{n!}, \qquad \text{ג)} \sum q^{n^2} z^n (|q| < 1), \qquad \text{ד)} \sum z^{n!}.$$

   b) Describe the convergence of the series in a) for $|z| = 1$.

4. **Problem 4.**

   a) For what values of z is

   $$\sum_{n=0}^{\infty} \left(\frac{z}{1+z}\right)^n$$

   convergent?

   b) Same question for

   $$\sum_{n=0}^{\infty} \frac{z^n}{1+z^{2n}}.$$

   Hint: Does there exists any difference between $|z| < 1$ and $|z| > 1$?

5. **Problem 5*.** The purpose of this exercise is to prove a weak version of the following Theorem of Levy and Steinitz: *If a series* $A = z_1 + z_2 + z_3 + ...$ *converges, but not absolutely, then there exists a line* l *such that for every* $s \in l$ *there exists a permutation* $\sigma(n)$ *of indices of* A*, such that the permuted series* $A' = a_{\sigma(1)} + a_{\sigma(2)} + a_{\sigma(3)} + ...$ *converges to* s.

   Definition. Let $\{r_n\}$, $\{s_m\}$ be two increasing sequences of natural numbers, such that $r_n < r_{n+1}$, $s_m < s_{m+1}$, $r_m \neq s_m$ $\forall m, n$, and $\cup_{n,m}(r_n \cup s_m) = \mathbf{N}$. Then two series

$$A_r := z_{r_1} + z_{r_2} + z_{r_3} + ... \qquad \text{and} \qquad A_s := z_{s_1} + z_{s_2} + z_{s_3} + ...$$

   are called *complemented parts* of a series $A = z_1 + z_2 + z_3 + ...$ (In other words, you color the terms of A in red and blue, say).

   If a permutation σ, $\sigma : \mathbf{N} \to \mathbf{N}$, of terms of A does not change the order inside A_r, A_s (if z_{r_m} appears before z_{r_n} and z_{s_m} appears before z_{s_n} for any m, n, $m < n$, after permutation), then one says that σ *moves* complemented parts A_r, A_s *with respect to each other.*

   Prove a)- f) and obtain the proof.

   a) The series $A_s(A_r)$ is convergent, provided A and $A_r(A_s)$ are convergent. If a permutation σ moves complemented parts A_r, A_s with respect to each other, then the sum $z_{\sigma(1)} + z_{\sigma(2)} + z_{\sigma(3)} + ...$ remains unchanged (is A).

   b) If a real A is convergent, but not absolutely, and if one of the complemented parts, say, A_r diverges to ∞, then another complemented part, A_s diverges to $-\infty$.

   c) If a real A is convergent, but not absolutely, and if all the terms of one of the complemented parts, say, A_r, have the same signs, then, using only the permutations that move the complemented parts, one can obtain any sum for the permuted series.

   d) If a series $|z_1| + |z_2| + |z_3| + ...$ diverges, then there exists a *direction of concentration* α, such that $\forall \epsilon > 0$, the series of absolute values of terms that are located in the angle $\alpha - \epsilon < \arg z < \alpha + \epsilon$ is divergent.

   e) If A is convergent, but not absolutely, and if a direction of concentration of A is a positive part of the real axes, prove that one can find $\{r_n\}$ such that $Re z_{r_1} + Re z_{r_2} + Re z_{r_3} + ... \to \infty$, but $Im z_{r_1} + Im z_{r_2} + Im z_{r_3} + ...$ is convergent.

   f) Prove the Theorem of Levy and Steinitz.