Complex Analysis, Spring 2011.

Instructor: Dmitry Ryabogin

Assignment V.

1. **Problem 1.**

   a) Find the value e^z for $z = \frac{\pi i}{2}, \frac{3\pi i}{4}$.

   b) For what values of z is e^z equal to $2, -1, i, -i/2, -1-i$?

   c) Find the real and imaginary parts of e^{e^z}.

   d) Determine the values of $2^i, i^i, (-1)^{2i}$.

   e) Express $\arctan w$ in terms of the logarithm.

2. **Problem 2.**

   a) Let $\Re a > 0$. The points of $\mathbf{C}$ can be divided into three sets, according to whether

$$\left|\frac{a-z}{\bar{a}+z}\right|$$

   is smaller, equal, or larger than 1. Describe these sets. Where does ∞ belong?

   Hint: Look at the sign of $-\Re((a+\bar{a})z)$.

   b) Let all zeros of the polynomial

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + ... + a_0$$

   to be located in the upper-half plane $\Im z > 0$. Denote $a_k = \alpha_k + i\beta_k$, $\alpha_k, \beta_k \in \mathbf{R}$. Show that

$$U(z) = \alpha_n z^n + \alpha_{n-1} z^{n-1} + ... + \alpha_0,$$
$$V(z) = \beta_n z^n + \beta_{n-1} z^{n-1} + ... + \beta_0,$$

   have only real zeros by proving א), ב), ג).

   א) Let

$$P(z) = U(z) + iV(z) = a_0(z-z_1)(z-z_2)...(z-z_n), \qquad a_0 \neq 0.$$

   Prove that

$$U(x) + iV(x) = U(x) - iV(x) \qquad \text{or} \qquad U(x) + iV(x) = -(U(x) - iV(x)),$$

   provided x is a root of $V(x) = 0$ or $U(x) = 0$.

   ב) Prove that

$$a_0(x-z_1)(x-z_2)...(x-z_n) = \pm\bar{a}_0(x-\bar{z}_1)(x-\bar{z}_2)...(x-\bar{z}_n).$$

   ג) Assume that, say, $\Im x > 0$. Observe that then $|x-z_k| < |x-\bar{z}_k|$, which is impossible. Why?

3. **Problem 3.**

   a) Find the set A_n of $z \in \mathbf{C}$ such that the n-th term, $z^n/n!$, $n = 0, 1, 2, ...$, of the series

   $$1 + \frac{z}{1!} + \frac{z^2}{2!} + ... + \frac{z^k}{k!} + ...$$

   exceeds the rest of the terms in absolute value, i.e., $|z^n/n!| \geq |z^k/k!|$ for $k \neq n$.

   b)* Let

   $$a_0 \neq 0, \qquad a_0 + a_1 z + ... + a_n z^n + ...$$

   be everywhere convergent and "never-ending" power series. Is it possible to divide the complex plane into disjoint open regions C_n, enclosed by concentric circles about the origin, such that in each C_n, $|a_n z^n| > |a_k z^k|$ for each $k \neq n$?

   Hint: No term could be maximal in absolute value for all z. When $|z|$ increases, the number $n = n(z)$, such that $|a_n z^n| \geq |a_k z^k|$ for $k \neq n$, can only increase.

4. **Problem 4.**

   a) Show how to define the "angles" in a triangle, bearing in mind that they should lie between 0 and π. With this definition, prove that the sum of the angles is π.

   b) Let $P(z)$ be a polynomial of degree n, $n \geq 2$, and let $a \neq b$ be such that $P(a) \neq P(b)$. Assume also that B is the *closed* set enclosed by two arcs, from which the segment ab is seen at the angle π/n. Prove that for any $\gamma \in [P(a), P(b)]$, there exists at least one solution $P(z) = \gamma$ with $z \in B$.

   Hint: Let z_k be a root of $P(z) - \gamma$,

   $$P(z) - \gamma = a_0(z - z_1)(z - z_2)...(z - z_n),$$

   where $\gamma = tP(a) + (1 - t)P(b)$, $t \in (0, 1)$. If $z_k \notin B$, then

   $$-\frac{\pi}{n} < \arg \frac{a - z_k}{b - z_k} < \frac{\pi}{n}.$$

   What can you say about

   $$\arg \frac{P(a) - \gamma}{P(b) - \gamma}\ ?$$