Complex Analysis, Spring 2011.

Instructor: Dmitry Ryabogin

Assignment VI.

1. **Problem 1.**

   a) Find a linear transformation (function) that transforms a triangle with vertices at $0, 1, i$ onto a similar triangle with vertices at $0, 2, 1+i$.

   b) Find a linear transformation with a fixed point $1+2i$ that transforms i into $-i$.

   c) For $w = 2z + 1 - 3i$ find a finite fixed point z_0, an angle θ of rotation about z_0, and a linear change of scale at z_0.

   d) Find a general form of a linear transformation that transforms the upper-half plane onto itself.

   e) Find a general form of a linear transformation that transforms the upper-half plane onto the lower-half plane.

2. **Problem 2.**

   a) Prove that the reflection $z \to \bar{z}$ is not a linear fractional transformation.

   b) If

   $$T_1 z = \frac{z+2}{z+3}, \qquad T_2 z = \frac{z}{z+1},$$

   find $T_1 T_2 z$, $T_2 T_1 z$ and $T_1^{-1} T_2 z$.

   c) Show that any fractional linear transformation which transforms the real axis into itself can be written with real coefficients.

3. **Problem 3.**

   a) Give a precise definition of a single-valued branch of $\sqrt{1+z} + \sqrt{1-z}$ in a suitable region, and prove that it is analytic.

   b) Same problem for $\log \log z$.

   c) Suppose that $f(z)$ is analytic and satisfies the condition $|f(z)^2 - 1| < 1$ in a region Ω. Show that either $\Re f(z) > 0$ or $\Re f(z) < 0$ throughout Ω.

4. **Problem 4.**

   a) Let $E \subset \mathbf{C}$ be a region with area $A(E)$ that can be evaluated as a double Riemann integral

   $$A(E) = \int\int_E dxdy.$$

   If f is a *conformal* mapping of an open set containing E, show that the area $A(E')$ of the image $E' = f(E)$ is given by

   $$A(E') = \int\int_E |f'(z)|^2 dxdy.$$

Hint: Use the change of variables formula in double integrals, then apply Cauchy-Riemann equations.

b) Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be one-to-one and analytic in the disc $D = \{z : |z| < 1\}$. Prove that $A(f(D)) \geq \pi$, and $A(f(D)) = \pi$ only provided $f(z) = z$.

Hint: Use a) together with $|f'(z)|^2 = f'(z)\overline{f'(z)}$ and

$$\int\int_D z^m \bar{z}^n \, dxdy = \int_0^1 r^{m+n+1} dr \int_0^{2\pi} e^{i(m-n)\theta} \, d\theta = 0 \qquad \text{for} \qquad m \neq n.$$