Complex Analysis, Spring 2011.

Instructor: Dmitry Ryabogin

Assignment VII.

1. **Problem 1.** Find a general form of a *linear* transformation that transforms:

   a) the upper-half plane onto the right-half plane;

   b) the right-half plane onto itself;

   c) the strip $0 < x < 1$ onto itself;

   d) the strip $-2 < y < 1$ onto itself;

   e) the strip bounded by lines $y = x$ and $y = x - 1$ on itself.

2. **Problem 2.** For an *inversion* $w = 1/z$ find the images of the following lines:

   a) a family of circles $x^2 + y^2 = ax$, $a \in \mathbf{R}$;

   b) a family of circles $x^2 + y^2 = by$, $b \in \mathbf{R}$;

   c) a family of parallel lines $y = x + b$, $b \in \mathbf{R}$;

   d) a family of lines $y = kx$, $k \in \mathbf{R}$;

   e) a family of lines passing through the given point $z_0 \neq 0$;

   f) $y = x^2$.

3. **Problem 3.**

   a) Find a *fractional linear* transformation which carries $0, i, -i$ into $1, -1, 0$.

   b) If a consecutive vertices z_1, z_2, z_3, z_4 of a quadrilateral lie on a circle, prove that

   $$|z_1 - z_3|\,|z_2 - z_4| = |z_1 - z_2|\,|z_3 - z_4| \; + \; |z_2 - z_3|\,|z_1 - z_4|.$$

   c) Find the fractional linear transformation which carries the circle $|z| = 2$ into $|z+1| = 1$, the point -2 into the origin, and the origin into i.

   d) Find the most general fractional linear transformation of the circle $|z| = R$ onto itself.

   Hint: Let $R = 1$, and let α is mapped into 0. Then $1/\bar{\alpha}$ is mapped into ∞. Why?

   e) Suppose that a fractional linear transformation carries one pair of concentric circles into another pair of concentric circles. Prove that the ratios of the radii must be the same. What conclusion can you make?

   Hint: Use symmetry to prove that $0 < |z| < \infty$ is mapped onto $0 < |w| < \infty$. Hence, the map is $w(z) = az$, or $w(z) = a/z$.

4. **Problem 4.**

Let $P(z) = \sum_{k=0}^{n} a_k z^k$ be a polynomial of degree n with all roots in the unit disc $|z| < 1$. Denote by $\tilde{P}(z) = \sum_{k=0}^{n} \bar{a}_k z^k$ a polynomial with conjugate coefficients, and let $P^*(z) = z^n \tilde{P}(1/z)$. Prove that all roots of $P(z) + P^*(z) = 0$ are on the unit circle $|z| = 1$.

Hint: Look at Problem 4 of Assignment 6. Let x be a root of $P + P^* = 0$. Then, $|P(x)| = |P^*(x)|$. But by 3 d) this is impossible...