# Complex Analysis, Spring 2011.

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# Assignment VIII.

1. **Problem 1.**

   a) If $z_1$, $z_2, z_3, z_4$ are points on the circle, show that $z_1$, $z_3$, $z_4$ and $z_2$, $z_3$, $z_4$ determine the same orientation iff $(z_1, z_2, z_3, z_4) > 0$.

   b) Prove that every reflection carries circles into circles.

   c) Suppose circles $C_1, C_2$ are both symmetric with respect to a line $l$, and $C_3$ is the reflection image of $C_1$ with respect to $C_2$. Then $C_3$ is also symmetric with respect to the line $l$, compare with the symmetry principle!

   **Hint**: Let $l$ be the real axis, and let the center of $C_2$ be on the real axis. What is the relation between $z$ and $z^*$?

   d) Reflect א) the imaginary axis, ב) the line $x = y$, and ג) the circle $|z| = 1$, in the circle $|z - 2| = 1$.

   **Hint**: In א) and ג) use the real axis as $l$. In ב) use $x + y = 2$.

2. **Problem 2.**

   a) Find the linear transformation which carries $|z| = 1$ and $|z - 1/4| = 1/4$ into concentric circles. What is the ratio of the radii?

   **Hint**: Use Problem 3 d) of Assignment VII with $R = 1$. If $0 \to -\beta$, $1/2 \to \beta$, what is $\beta$?

   b) Same problem for $|z| = 1$ and $x = 2$.

   **Hint**: Use $w = 1/z$.

   c) Find all circles which are orthogonal to $|z| = 1$ and $|z - 1| = 4$.

   **Hint**: Use the inversion that transforms the given circles into concentric ones.

3. **Problem 3.**

   a) Find the fixed points of the linear transformations

   $$\text{א)}\ w = \frac{z}{2z-1}, \qquad \text{ב)}\ w = \frac{2z}{3z-1}, \qquad \text{ג)}\ w = \frac{3z-4}{z-1}, \qquad \text{ד)}\ w = \frac{z}{2-z}.$$

   Is any of transformations elliptic, hyperbolic, parabolic?

   b) Suppose that the coefficients of the transformation $Sz = (az + b)/(cz + d)$ are normalized by $ad - bc = 1$. Show that $S$ is elliptic iff $-2 < a + d < 2$, parabolic if $a + d = \pm 2$, hyperbolic if $a + d < -2$ or $> 2$.

4. **Problem 4.**

a) Show that a linear transformation which satisfies $S^n z = z$ for some integer $n$ is necessary elliptic.

**Hint**: Consider the number of fixed points. If there are two distinct finite fixed points $a$ and $b$, say. Then,

$$\frac{z-a}{z-b} = \frac{S^n z - a}{S^n z - b} = k\frac{S^{n-1}z - a}{S^{n-1}z - b} = k^2 \frac{S^{n-2}z - a}{S^{n-2}z - b} = ...$$

If $S$ has only one fixed point, consider $Tz = 1/(z-a) + a$ and observe $TST^{-1}$ has only $\infty$ as a fixed point. Hence, $TST^{-1}z = cz + d$, and $c = 1$.

b) If $S$ is hyperbolic or loxodromic, show that $S^n z$ converges to a fixed point as $n \to \infty$, the same for all $z$, except when $z$ coincides with the other fixed point, (the limit is the *attractive*, the other the *repellent* fixed point). What happens when $n \to -\infty$? What happens in the parabolic case?

**Hint**: Let at first two fixed points $z_1$, $z_2$ be finite. Passing to "new coordinates", we can write $v = L(\xi) = K\xi$, where

$$v = \frac{w - z_1}{w - z_2}, \qquad \xi = \frac{z - z_1}{z - z_2}.$$

Hence, $L(L(...L(\xi))) = K^n \xi ...$