Complex Analysis, Spring 2011.

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Assignment IX.

1. **Problem 1.**

   a) Map the common part of the discs $|z| < 1$ and $|z-1| < 1$ on the inside of the unit circle.

   Hint: Use $w = (z-a)(z-b)$.

   b) Map the region between $|z| = 1$ and $|z - 1/2| = 1/2$ on the inside of the unit circle.

   Hint: Use $w = 1/(z-a)$.

   c) Map the complement of the arc $|z| = 1$, $y \geq 0$ on the outside of the unit circle.

   Hint: Map the region onto the complement of the segment $\Im z = 0$, $-1 \leq \Re z \leq 1$, (the wedge).

2. **Problem 2.**

   a) Map the outside of the parabola $y^2 = 2px$ on the disc $|w| < 1$ so that $z = 0$ and $z = -p/2$ correspond to $w = 1$ and $w = 0$.

   b) Map the inside of the right-hand branch of the hyperbola $x^2 - y^2 = a^2$ on the unit disc $|w| < 1$ so that the focus corresponds to $w = 0$ and the vertex to $w = -1$.

   c) Map the inside of the lemniscate $|z^2 - a^2| = \rho^2$, $\rho > a$, on the disc $|w| < 1$.

   d) Map the outside of the ellipse $(x/a)^2 + (y/b)^2 = 1$ onto $|w| < 1$.

   Hint: Find s such that $r + 1/r = as$, $1/r - r = bs$.

3. **Problem 3.**

   a) Let a fractional linear transformation carrying a unit disc into itself be

   $$w = e^{i\alpha}\frac{z-a}{1-\bar{a}z}, \qquad |a| < 1, \qquad a = |a|e^{i\lambda}.$$

   Prove that it can be hyperbolic, parabolic and elliptic, but not loxodromic.

   Hint: Compute K and use $\cos\alpha = (e^{i\alpha} + e^{-i\alpha})/2$, $\sin\alpha = (e^{i\alpha} - e^{-i\alpha})/(2i)$.

   b) Describe a for which each of these cases occurs.

   c) Find fixed points and bring the transformation to the normal form.

   Hint: Put $|a| = \sin(\alpha/2)\sin\beta$ for some $\beta \in (0, \pi)$ when w is elliptic. If w is hyperbolic, put $|a|\sin\beta = \sin(\alpha/2)$.

   d) Find $\arg w(e^{i\phi}) = \theta(\phi)$.

   e) Find $w'(0)$, $w'(a)$.

   f) What part of the unit disc is dilated, what part is shrinking?

   g) Find $\max|w'(z)|$, $\min|w'(z)|$ for $|z| \leq 1$.