Detecting symmetry in star bodies

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A R T I C L E   I N F O

A B S T R A C T

We use Fourier transform techniques to prove a result on detecting symmetry in convex and star bodies with the help of conical sections. Our methods also allow us to give a new proof of the well-known theorem of Makai, Martini and Ódor about maximal hyperplane sections passing through the same point.

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1. Introduction

Let K be a convex body in $\mathbb{R}^n$, i.e. a compact convex set with a non-empty interior. We say that K is *origin-symmetric* if $K = -K$. The presence of origin-symmetry is an essential assumption in various problems. Many results that hold for origin-symmetric convex bodies fail in the absence of the symmetry condition. For example, origin-symmetric convex bodies are uniquely determined by the volumes of their projections or central sections, while this is not true for general convex bodies; see [1]. Thus, detecting symmetry in convex bodies is one of the fundamental questions in convex geometry and geometric tomography. For some results in this direction the reader is referred to [2–10]; see also [11] for open problems.

In this paper we suggest a new method of detecting symmetry. Let K be a star body and let $C(\xi, z)$ be the cone $\{x \in \mathbb{R}^n : x \cdot \xi = |x|z\}$, where $\xi \in S^{n-1}$, $z \in (-1, 1)$, and $x \cdot \xi = x_1\xi_1 + x_2\xi_2 + \cdots + x_n\xi_n$ is the usual inner product in $\mathbb{R}^n$. In this notation, z is the cosine of the angle between x and ξ. For $z \in (-1, 1)$, we define the *conical section function* $C_{K,\xi}(z)$ by
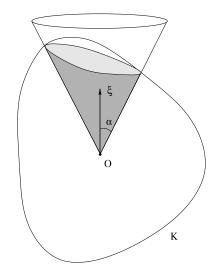
$$C_{K,\xi}(z) = \text{vol}_{n-1}(K \cap C(\xi, z)).$$

In the picture below, the shaded part represents the intersection $K \cap C(\xi, z)$, and $\alpha = \arccos z$.

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Clearly, if K is an origin-symmetric star body, then for each ξ the function $C_{K,\xi}(z)$ is an even function of z, and therefore has a critical point at $z = 0$. In this paper we show that the converse statement is also true.

Theorem 1.1. *Let K be a C^1 star body in $\mathbb{R}^n$. Assume that for each $\xi \in S^{n-1}$ the function $C_{K,\xi}(z)$ has a critical point at $z = 0$. Then the body K is origin-symmetric.*

This theorem is an analog of the result by Makai et al. [9], which can be stated as follows.

Theorem 1.2. *Let K be a C^1 star body in $\mathbb{R}^n$. If for every $\xi \in S^{n-1}$ the function $A_{K,\xi}(t)$ has a critical point at $t = 0$, then K is origin-symmetric.*

Here, $A_{K,\xi}(t)$ is the *parallel section function* defined by

$$A_{K,\xi}(t) = \mathrm{vol}_{n-1}(K \cap (\xi^\perp + t\xi)), \quad t \in \mathbb{R},$$

and $\xi^\perp = \{x \in \mathbb{R}^n : x \cdot \xi = 0\}$ is the hyperplane passing through the origin and orthogonal to the vector ξ.

Makai et al. proved Theorem 1.2 in the class of convex bodies, in which case the C^1-smoothness assumption can be dropped. Using the same reasoning (see [9, Lemma 3.5] for details), it can be shown that Theorem 1.1 also holds true for convex bodies without the smoothness assumption.

The techniques that we use in this paper were developed by Koldobsky (see [12]) and are based on the Fourier transform of distributions. Using these methods we also give a new and short proof of Theorem 1.2.

The study of properties of convex bodies using the information about the areas of their planar sections is the classical problem of geometric tomography. However, a natural question of what happens if plane sections are replaced by sections by other surfaces has not been studied well. In this note we make a step in this direction by considering sections by conical surfaces. In fact, a lot of problems like determination of symmetric bodies by central sections, the Busemann–Petty problem and others can be asked in the setting of surfaces; see [13] for some results. These problems may be quite difficult, but they belong to a new interesting direction.

2. Notation and auxiliary results

A *body* is a compact set equal to the closure of its interior. If K is a body containing the origin in its interior and star-shaped with respect to the origin, its *radial function* is defined by

$$\rho_K(x) = \max\{a \geq 0 : ax \in K\},$$

for $x \in \mathbb{R}^n \setminus \{0\}$. If $\xi \in S^{n-1}$, then $\rho_K(\xi)$ is the distance from the origin to the point on the boundary in the direction of ξ. A body K is called a *star body* if its radial function is positive and continuous. We say that a star body K is of class C^k if $\rho_K \in C^k(S^{n-1})$. The *Minkowski functional* of a star body $K \subset \mathbb{R}^n$ is defined by

$$\|x\|_K = \min\{a \geq 0 : x \in aK\}, \quad x \in \mathbb{R}^n.$$

It easy to see that $\rho_K(x) = \|x\|_K^{-1}$ for $x \in \mathbb{R}^n \setminus \{0\}$.

The main tool that we use in this paper is the Fourier transform of distributions. For the background information, the reader is referred to the books by Gelfand and Shilov [14] and by Koldobsky [12].

Let $\mathscr{S}(\mathbb{R}^n)$ be the Schwartz space of rapidly decreasing infinitely differentiable functions on $\mathbb{R}^n$. Elements of this space are referred to as test functions. *Distributions* are the elements of the dual space, $\mathscr{S}'(\mathbb{R}^n)$, of linear continuous functionals on $\mathscr{S}(\mathbb{R}^n)$. The action of a distribution f on a test function ϕ is denoted by $\langle f, \phi \rangle$.

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Let $\phi \in \mathcal{S}(\mathbb{R})$. The *fractional derivative* of the function ϕ of order $q \in \mathbb{C}$ at zero is defined as follows

$$\phi^{(q)}(0) = \left\langle \frac{t_+^{-1-q}}{\Gamma(-q)}, \phi(t) \right\rangle,$$

where $t_+ = \max\{0, t\}$.

If $\Re q < 0$, then the function t^{-1-q} is locally integrable and the above fractional derivative is equal to

$$\phi^{(q)}(0) = \frac{1}{\Gamma(-q)} \int_0^\infty t^{-1-q} \phi(t) dt.$$

This integral can be written in the following form; see [12, Sec. 2.5 and 2.6] for details:

$$\phi^{(q)}(0) = \frac{1}{\Gamma(-q)} \int_0^1 t^{-1-q} \left(\phi(t) - \phi(0) - \cdots - \phi^{(m-1)}(0) \frac{t^{m-1}}{(m-1)!} \right) dt$$
$$+ \frac{1}{\Gamma(-q)} \int_1^\infty t^{-1-q} \phi(t) dt + \frac{1}{\Gamma(-q)} \sum_{k=0}^{m-1} \frac{\phi^{(k)}(0)}{k!(k-q)}.$$

Note that the latter expression makes sense for q with $-1 < \Re q < m, q \neq 0, 1, \ldots, m-1$, and this is how $\phi^{(q)}(0)$ is defined for these values of q.

If $k \geq 0$ is an integer, we define the fractional derivative of the order k as the limit of the latter expression as $q \to k$, then we get

$$\phi^{(k)}(0) = (-1)^k \frac{d^k}{dt^k} \phi(t) \Big|_{t=0},$$

i.e. fractional derivatives of integral orders coincide up to a sign with ordinary derivatives. Thus defined, $\phi^{(q)}(0)$ is an entire function of the variable $q \in \mathbb{C}$. Note that the fractional derivatives $\phi^{(q)}(0)$ can also be defined if ϕ is a continuous function with compact support and sufficiently differentiable in a neighborhood of zero.

The Fourier transform of $\phi \in \mathcal{S}(\mathbb{R}^n)$ is defined by

$$\hat{\phi}(x) = \int_{\mathbb{R}^n} \phi(y) e^{-ix \cdot y} dy \quad \text{for } x \in \mathbb{R}^n.$$

The Fourier transform of a distribution f is defined by its action on a test function as follows:

$$\langle \hat{f}, \phi \rangle = \langle f, \hat{\phi} \rangle,$$

for any test function ϕ.

Our main tool is the Fourier transform of homogeneous distributions on $\mathbb{R}^n$. For $f \in C^\infty(S^{n-1})$ and $p \in \mathbb{C}$, we denote by f_p the homogeneous degree $-n+p$ extension of f to $\mathbb{R}^n \setminus \{0\}$. Thus,

$$f_p(x) = |x|^{-n+p} f\left(\frac{x}{|x|} \right) \quad \text{for } x \neq 0.$$

Formulas for the Fourier transform of f_p were obtained in the case of even functions in [15] (see also [12]) and in the general case in [16]. We will need the following auxiliary function. For $f \in C(S^{n-1})$ and $\xi \in S^{n-1}$, the function F_ξ is defined by

$$F_\xi(t) = (1 - t^2)^{(n-3)/2} \int_{S^{n-1} \cap \xi^\perp} f(t\,\xi + \sqrt{1-t^2}\,\zeta)\, d\zeta, \quad t \in (-1, 1).$$

If Φ is an integrable function on $[-1, 1]$, then

$$\int_{-1}^1 \Phi(t) F_\xi(t) dt = \int_{S^{n-1}} \Phi(\theta \cdot \xi) f(\theta) d\theta;$$

cf. [4] or [17].

If $0 < \Re p < 1$, then the Fourier transform of f_p is a homogeneous function of degree $-p$ on $\mathbb{R}^n \setminus \{0\}$ given by

$$\hat{f}_p(x) = \Gamma(p) \cos \frac{p\pi}{2} \int_{S^{n-1}} |x \cdot \theta|^{-p} f(\theta)\, d\theta - i\Gamma(p) \sin \frac{p\pi}{2} \int_{S^{n-1}} |x \cdot \theta|^{-p} \mathrm{sgn}\,(x \cdot \theta) f(\theta)\, d\theta. \tag{1}$$

Using regularization, as in the case of fractional derivatives, one can obtain formulas for $\hat{f}_p$ when $\Re p \geq 1$. In particular, for $p = 1, 3, \ldots$, we have

$$\hat{f}_p(\xi) = -i(-1)^{(p-1)/2}(p-1)!\left(\int_{-1}^{1}|t|^{-p}\operatorname{sgn} t\left(F_\xi(t) - \sum_{j=0}^{p-1}\frac{t^j}{j!}F_\xi^{(j)}(0)\right)dt + \sum_{\substack{0\leq j\leq p-1 \\ j\,\text{odd}}}\frac{2}{j!(1+j-p)}F_\xi^{(j)}(0)\right)$$

$$+ (-1)^{(p-1)/2}\pi F_\xi^{(p-1)}(0),$$

whereas, for $p = 2, 4, \ldots$,

$$\hat{f}_p(\xi) = (-1)^{p/2}(p-1)!\left(\int_{-1}^{1}|t|^{-p}\left(F_\xi(t) - \sum_{j=0}^{p-1}\frac{t^j}{j!}F_\xi^{(j)}(0)\right)dt + \sum_{\substack{0\leq j\leq p-1 \\ j\,\text{even}}}\frac{2}{j!(1+j-p)}F_\xi^{(j)}(0)\right) + i(-1)^{p/2}\pi F_\xi^{(p-1)}(0).$$

Remark 1. It follows from the proof of these formulas that assumption $f \in C^\infty\left(S^{n-1}\right)$ can be relaxed. For example, if $0 < \Re p < 1$, formula (1) remains valid for $f \in C\left(S^{n-1}\right)$. If p is an integer, then it is enough to require that $f \in C^p\left(S^{n-1}\right)$.

3. Proofs of results

Proof of Theorem 1.1. Consider the following auxiliary function

$$G_{K,\xi}(z) = \begin{cases} (1-z^2)^{-1/2} \cdot \operatorname{vol}_{n-1}(K \cap C(\xi, z)), & |z| < 1, \\ 0, & |z| \geq 1. \end{cases}$$

By our assumption, $G'_{K,\xi}(0) = 0$ for all $\xi \in S^{n-1}$.

First, we establish the following formula:

$$G_{K,\xi}(z) = \begin{cases} \dfrac{1}{n-1}(1-z^2)^{(n-3)/2}\displaystyle\int_{S^{n-1}\cap\xi^\perp}\rho_K^{n-1}(z\xi + \sqrt{1-z^2}\theta)d\theta, & |z| < 1, \\ 0, & |z| \geq 1. \end{cases} \tag{2}$$

In order to prove (2), we compute $\operatorname{vol}_{n-1}(K \cap C(\xi, z))$ using the idea from [13]. Denote by $(K \cap C(\xi, z))|\xi^\perp$ the orthogonal projection of $K \cap C(\xi, z)$ onto the hyperplane $\xi^\perp$. Let $\alpha = \arccos z$. The cosine of the angle between the generating lines of the cone $C(\xi, z)$ and the hyperplane $\xi^\perp$ equals $\sin \alpha = (1 - z^2)^{1/2}$. Projecting $K \cap C(\xi, z)$ onto the hyperplane $\xi^\perp$, we get

$$\operatorname{vol}_{n-1}\left((K \cap C(\xi, z))|\xi^\perp\right) = (1 - z^2)^{1/2}\operatorname{vol}_{n-1}(K \cap C(\xi, z)). \tag{3}$$

Let $\theta \in S^{n-1} \cap \xi^\perp$. Then the radius of $(K \cap C(\xi, z))|\xi^\perp$ in the direction of θ is equal to $\sin \alpha \cdot \rho_K(\cos \alpha \xi + \sin \alpha \theta)$. Therefore, computing the volume of the projection in polar coordinates, we get

$$\operatorname{vol}_{n-1}\left((K \cap C(\xi, z))|\xi^\perp\right) = \frac{1}{n-1}\int_{S^{n-1}\cap\xi^\perp}(\sin\alpha)^{n-1}\rho_K^{n-1}(\cos\alpha\xi + \sin\alpha\theta)\,d\theta.$$

Combining the latter formula with (3), we get

$$\operatorname{vol}_{n-1}(K \cap C(\xi, z)) = \frac{(1-z^2)^{(n-2)/2}}{n-1}\int_{S^{n-1}\cap\xi^\perp}\rho_K^{n-1}(z\xi + \sqrt{1-z^2}\theta)\,d\theta.$$

Thus, formula (2) is proved.

Now we compute fractional derivatives of order q at $z = 0$ for the function $G_{K,\xi}(z)$. Our goal is to show that

$$G_{K,\xi}^{(q)}(0) = \frac{\cos(q\pi/2)}{2\pi(n-1)}\left(\|x\|_K^{-n+1}|x|^q + \|-x\|_K^{-n+1}|x|^q\right)^\wedge(\xi)$$

$$- \frac{i\sin(q\pi/2)}{2\pi(n-1)}\left(\|x\|_K^{-n+1}|x|^q - \|-x\|_K^{-n+1}|x|^q\right)^\wedge(\xi). \tag{4}$$

First we assume that $-1 < \Re q < 0$ and then use the analytic extension. We have

$$G_{K,\xi}^{(q)}(0) = \frac{1}{\Gamma(-q)}\int_0^\infty z^{-1-q}G_{K,\xi}(z)dz$$

$$= \frac{1}{2\Gamma(-q)}\int_{\mathbb{R}}\left(|z|^{-1-q} + |z|^{-1-q}\operatorname{sgn} z\right)G_{K,\xi}(z)dz$$

$$= \frac{1}{2(n-1)\Gamma(-q)} \int_{-1}^{1} \left(|z|^{-1-q} + |z|^{-1-q}\mathrm{sgn}\, z\right) (1-z^2)^{(n-3)/2} \int_{S^{n-1}\cap\xi^{\perp}} \rho_K^{n-1}(z\xi + \sqrt{1-z^2}\theta)d\theta\, dz$$

$$= \frac{1}{2(n-1)\Gamma(-q)} \int_{S^{n-1}} \left(|x\cdot\xi|^{-1-q} + |x\cdot\xi|^{-1-q}\mathrm{sgn}\,(x\cdot\xi)\right)\rho_K^{n-1}(x)\, dx$$

$$= \frac{1}{4(n-1)\Gamma(-q)} \int_{S^{n-1}} |x\cdot\xi|^{-1-q} \left(\rho_K^{n-1}(x) + \rho_K^{n-1}(-x)\right)\, dx$$

$$+ \frac{1}{4(n-1)\Gamma(-q)} \int_{S^{n-1}} |x\cdot\xi|^{-1-q}\mathrm{sgn}\,(x\cdot\xi)\left(\rho_K^{n-1}(x) - \rho_K^{n-1}(-x)\right)\, dx.$$

The latter equality allows us to consider $G_{K,\xi}^{(q)}(t)$ as a function of $\xi \in \mathbb{R}^n \setminus \{0\}$, and write it in terms of the Fourier transform. By virtue of formula (1),

$$G_{K,\xi}^{(q)}(0) = -\frac{1}{4(n-1)\Gamma(-q)\Gamma(q+1)\sin(q\pi/2)} \left(\|x\|_K^{-n+1}|x|^q + \|-x\|_K^{-n+1}|x|^q\right)^{\wedge}(\xi)$$

$$+ \frac{i}{4(n-1)\Gamma(-q)\Gamma(q+1)\cos(q\pi/2)} \left(\|x\|_K^{-n+1}|x|^q - \|-x\|_K^{-n+1}|x|^q\right)^{\wedge}(\xi).$$

Since

$$\Gamma(-q)\Gamma(q+1) = -\frac{\pi}{\sin q\pi},$$

(see [12, p. 31]), formula (4) is proved for the range $-1 < \Re q < 0$. It can be extended to $-1 < \Re q < n-1$ via an analytic continuation argument (see [12, pp. 60–61] and [16, Theorem 3.1] for details). In particular, for $q = 1$ we obtain the following formula:

$$G_{K,\xi}^{(1)}(0) = -\frac{i}{2\pi(n-1)} \left(\|x\|_K^{-n+1}|x| - \|-x\|_K^{-n+1}|x|\right)^{\wedge}(\xi).$$

Finally, we use the condition $G_{K,\xi}'(0) = 0$ for all $\xi \in S^{n-1}$ to get

$$\left(\|x\|_K^{-n+1}|x| - \|-x\|_K^{-n+1}|x|\right)^{\wedge}(\xi) = 0, \quad \forall \xi \in S^{n-1}.$$

Due to homogeneity, the latter formula holds for all $\xi \in \mathbb{R}^n \setminus \{0\}$.

Inverting the Fourier transform we get

$$\|x\|_K^{-n+1}|x| - \|-x\|_K^{-n+1}|x| = 0, \quad x \in \mathbb{R}^n \setminus \{0\},$$

which means that

$$\|x\|_K = \|-x\|_K, \quad x \in \mathbb{R}^n,$$

i.e. the body K is origin-symmetric. $\square$

Proof of Theorem 1.2. As above, our goal is to derive a formula for fractional derivatives of the function $A_{K,\xi}(t)$ at $t = 0$ and then use the condition $A_{K,\xi}'(0) = 0$. The following calculations are known in the class of origin-symmetric convex bodies; see [15] or [12]. We will extend these results to cover the general case.

Let $-1 < \Re q < 0$. Using the definition of fractional derivatives, the Fubini theorem, and integration in polar coordinates we get

$$A_{K,\xi}^{(q)}(0) = \frac{1}{\Gamma(-q)} \int_0^{\infty} z^{-1-q} A_{K,\xi}(z)dz$$

$$= \frac{1}{2\Gamma(-q)} \int_{\mathbb{R}} \left(|z|^{-1-q} + |z|^{-1-q}\mathrm{sgn}\, z\right) A_{K,\xi}(z)dz$$

$$= \frac{1}{2\Gamma(-q)} \int_{\mathbb{R}} \left(|z|^{-1-q} + |z|^{-1-q}\mathrm{sgn}\, z\right) \int_{(x,\xi)=z} \chi\left(\|x\|_K\right) dx\, dz$$

$$= \frac{1}{2\Gamma(-q)} \int_K \left(|x\cdot\xi|^{-1-q} + |x\cdot\xi|^{-1-q}\mathrm{sgn}\,(x\cdot\xi)\right)\, dx$$

$$= \frac{1}{2\Gamma(-q)} \int_{S^{n-1}} \left(|x\cdot\xi|^{-1-q} + |x\cdot\xi|^{-1-q}\mathrm{sgn}\,(x\cdot\xi)\right) \int_0^{\rho_K(\theta)} r^{n-1}r^{-1-q}dr\, d\theta$$

$$= \frac{1}{2(n-1-q)\Gamma(-q)} \int_{S^{n-1}} \left(|x\cdot\xi|^{-1-q} + |x\cdot\xi|^{-1-q}\mathrm{sgn}\,(x\cdot\xi)\right) \rho_K^{n-1-q}(\theta)\, d\theta$$

$$= \frac{1}{4(n-1-q)\Gamma(-q)} \int_{S^{n-1}} |x \cdot \xi|^{-1-q} \left(\rho_K^{n-1-q}(\theta) + \rho_K^{n-1-q}(-\theta) \right) d\theta$$

$$+ \frac{1}{4(n-1-q)\Gamma(-q)} \int_{S^{n-1}} |x \cdot \xi|^{-1-q} \operatorname{sgn}(x \cdot \xi) \left(\rho_K^{n-1-q}(\theta) - \rho_K^{n-1-q}(-\theta) \right) d\theta.$$

The latter equality allows us to consider $A_{K,\xi}^{(q)}(0)$ as a function of $\xi \in \mathbb{R}^n$, and write it in terms of Fourier transforms using formula (1) (cf. (4)),

$$A_{K,\xi}^{(q)}(0) = \frac{\cos(q\pi/2)}{2\pi(n-1-q)} \left(\|x\|_K^{-n+1+q} + \|-x\|_K^{-n+1+q} \right)^{\wedge}(\xi)$$

$$- \frac{i\sin(q\pi/2)}{2\pi(n-1-q)} \left(\|x\|_K^{-n+1+q} - \|-x\|_K^{-n+1+q} \right)^{\wedge}(\xi).$$

By the analytic extension argument mentioned above, the formula can be extended to $-1 < \Re q < n-1$. Putting $q = 1$ in the latter formula and using the condition $A_{K,\xi}'(0) = 0, \forall \xi \in S^{n-1}$, we get

$$\left(\|-x\|_K^{-n+2} - \|x\|_K^{-n+2} \right)^{\wedge}(\xi) = 0, \quad \forall \xi \in S^{n-1}.$$

Therefore, $\|-x\|_K^{-n+2} - \|x\|_K^{-n+2} = 0$, for $x \in \mathbb{R}^n \setminus \{0\}$, i.e. the body K is origin-symmetric. $\quad\square$

Acknowledgments

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