SINGULAR INTEGRAL OPERATORS GENERATED BY WAVELET TRANSFORMS

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Let $g \in L^1(\mathrm{R}^n)$, $g_t(x) = t^{-n} g(x/t)$. If $\int g = 0$, then g is called the wavelet function, and the convolution operator $f \to f * g_t$ is called the wavelet transform of f generated by the wavelet g. For a large class of functions g and $f \in L^p(\mathrm{R}^n)$, $1 < p < \infty$, it is shown that $\int_\varepsilon^\rho (f * g_t)(x) dt/t$ converges as $\varepsilon \to 0$ and $\rho \to \infty$ in the L^p-norm and in the a.e. sense to a limit $I(f,g) = cf + Tf$, where $c \equiv c(g) = const$, and $T \equiv T(g)$ is the Calderón-Zygmund singular integral operator. The particular case $T \equiv 0$ corresponds to Calderón's reproducing formula. Each "rough" singular integral operator $T_\theta f = p.v.\ |x|^{-n}\theta(x/|x|) * f$, with $\theta \in H^1(\Sigma_{n-1})$ (the Hardy space on the unit sphere) can be represented as $I(f,g)$ with a suitable wavelet g. A new proof of the L^p-boundedness of T_θ, $\theta \in H^1(\Sigma_{n-1})$, is given.

1 INTRODUCTION.

The present article is motivated by the following problems, which are closely related to each other.

1. If $f \in L^p(\mathrm{R}^n)$, $1 < p < \infty$, and g is an integrable radial function, then ([12], [13])

$$\lim_{\substack{\varepsilon \to 0 \\ \rho \to \infty}}^{(L^p)} \int_\varepsilon^\rho \frac{f * g_t}{t}\, dt = c_g f, \qquad c_g = \int_{\mathrm{R}^n} g(x) \log \frac{1}{|x|} dx, \tag{1.1}$$

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provided

$$\int_{\mathrm{R}^n} g(x)dx = 0 \qquad \text{and} \qquad \int_{\mathrm{R}^n} |g(x)\log|x||dx < \infty. \tag{1.2}$$

The first equality in (1.1) is a variation of the Calderón reproducing formula ([5], p. 8). What can one say about the limit in (1.1) if g is not radial? For which nonradial functions g does the reproducing formula (1.1) still hold?

2. If g belongs to the Schwartz space $S(\mathrm{R}^n)$ and $\int g = 0$, then ([15], p. 45) $\int_\varepsilon^\rho g_t dt/t$ tends (as $\varepsilon \to 0$ and $\rho \to \infty$) to the distribution of the form $c\delta + p.v.K$, where $c = c(g) = const$, δ is the Dirac delta function, and K is a certain Calderón-Zygmund kernel, having the form $\Omega(y/|y|)/|y|^n$ with the characteristic Ω depending on g. The expected L^p-analogue of this statement reads :

$$\overset{(L^p)}{\lim_{\substack{\varepsilon\to 0\\ \rho\to\infty}}} \int_\varepsilon^\rho \frac{f * g_t}{t}\, dt = cf + Tf, \quad f \in L^p, \tag{1.3}$$

$T \equiv T(g)$ being a singular integral operator defined by $Tf = p.v.\ K * f$. The problem is to prove (1.3) and to find a possibly large class of integrable functions g for which (1.3) holds.

In the case $n = 1$ these problems were studied in ([11]; Sec. 12). One should mention the earlier papers by M. J. Fisher [6, 7], in which the L^p-convergence of integrals similar to that in (1.1) was studied in the more general setting for functions f defined on a Hilbert space H. The results from [6, 7], reformulated for the special case $H = \mathrm{R}^n$, are less general than those presented below. For example, we interpret the limit of $\int_\varepsilon^\rho (f * g_t)dt/t$ not only in the L^p-norm but also in the "almost everywhere" sense. Furthermore, the integral c_g (see (1.1)) in our consideration can be regarded in the sense of duality between $BMO(\mathrm{R}^n)$ (we recall that $\log(1/|x|) \in BMO(\mathrm{R}^n)$) and the Hardy space $H^1(\mathrm{R}^n)$. The "standard" case, when c_g is represented as an absolutely convergent integral, is also considered. Our proofs seem to be simpler than those in [6, 7], and employ different ideas.

In order to state main results we denote: $\Sigma_{n-1} = \{x \in \mathrm{R}^n : |x| = 1\}$. The Hilbert transform $H_{y'}f$ of f in the direction $y' = y/|y| \in \Sigma_{n-1}$, and the Riesz transforms $R_j f$ are defined by

$$(H_{y'}f)(x) = \frac{1}{\pi i}\, p.v. \int_{-\infty}^{\infty} \frac{f(x - ty')}{t}\, dt, \tag{1.4}$$

$$(R_j f)(x) = \frac{\Gamma\left((n+1)/2\right)}{i\pi^{(n+1)/2}}\, p.v. \int_{\mathrm{R}^n} \frac{x_j - y_j}{|x - y|^{n+1}} f(y)dy\ , \quad j = 1, \ldots, n. \tag{1.5}$$

The notation $H^1(\mathrm{R}^n)$ and $H^1(\Sigma_{n-1})$ is used for the coresponding Hardy spaces [3, 4, 10]. The letter c designates a constant which can be different at each occurence. We will use the notation $I_{\varepsilon,\rho}(f,g)(x) = \int_\varepsilon^\rho (f * g_t)(x)dt/t$, $0 < \varepsilon < \rho < \infty$.

Theorem 1.1. *Let $f \in L^p(\mathrm{R}^n)$, $1 < p < \infty$, $g \in H^1(\mathrm{R}^n)$. Then*

$$\| \sup_{0<\varepsilon<\rho<\infty} I_{\varepsilon,\rho}(f,g)| \|_p \le c\|g\|_{H^1}\|f\|_p, \tag{1.6}$$

and the limit $\lim\limits_{\substack{\varepsilon\to 0\\ \rho\to\infty}} I_{\varepsilon,\rho}(f,g)$ *exists in the L^p-norm and in the a.e. sense.*

In order to state our next results, we need the Calderón-Zygmund singular integral operators defined by

$$(T_\theta f)(x) = \lim_{\substack{\varepsilon\to 0\\ \rho\to\infty}} (T_\theta^{\varepsilon,\rho} f)(x), \quad (T_\theta^{\varepsilon,\rho} f)(x) = \int\limits_{\varepsilon<|y|<\rho} f(x-y)\frac{\theta(y')}{|y|^n}dy. \tag{1.7}$$

Theorem 1.2. *Suppose that $\theta \in H^1(\Sigma_{n-1})$, $\int_{\Sigma_{n-1}} \theta(x')dx' = 0$, $f \in L^p(\mathrm{R}^n)$, $1 < p < \infty$. Then* $\| \sup\limits_{0<\varepsilon<\rho<\infty} |T_\theta^{\varepsilon,\rho} f|\|_p \le c\ \|\theta\|_{H^1(\Sigma_{n-1})}\|f\|_p$, *and the limit* $T_\theta f = \lim\limits_{\substack{\varepsilon\to 0\\ \rho\to\infty}} T_\theta^{\varepsilon,\rho} f$ *exists in the L^p-norm and in the a.e. sense.*

This theorem was first proved by Calderón and Zygmund [1] for θ belonging to $L\log^+ L(\Sigma_{n-1})$. The more general case $\theta \in H^1(\Sigma_{n-1})$, was studied in [2, 3, 8-10, 17]. In Section 4 we give an alternative proof of Theorem 1.2 based on Theorem 1.1.

Theorem 1.3. *Assume that $g(x)$ is a measurable function satisfying* (1.2) *and such that $\Omega(x') = \int_0^\infty r^{n-1} g(rx')dr \in H^1(\Sigma_{n-1})$. For $f \in L^p(\mathrm{R}^n)$, $1 < p < \infty$, the following statements hold*:

$$(i) \qquad I(f,g) \equiv \overset{(L^p)}{\lim_{\substack{\varepsilon\to 0\\ \rho\to\infty}}} I_{\varepsilon,\rho}(f,g) = c_g f + T_\Omega f, \quad c_g = \int_{\mathrm{R}^n} g(x)\log\frac{1}{|x|}dx, \tag{1.8}$$

where $T_\Omega f = \overset{(L^p)}{\lim\limits_{\substack{\varepsilon\to 0\\ \rho\to\infty}}} T_\Omega^{\varepsilon,\rho} f$ *(cf.* (1.7)).

(ii) If, moreover,

$$\int\limits_{|x|<1/2} |g(x)|\,|\log|x|\,|^{1+\delta}dx < \infty \ \ for\ some\ \ \delta > 0, \tag{1.9}$$

then $\| \sup\limits_{0<\varepsilon<\rho<\infty} |I_{\varepsilon,\rho}(f,g)|\,\|_p \le c\ \|f\|_p$, $c = c(g)$, *and (i) holds with convergence interpreted in the a.e. sense.*

This theorem can be extended to $g \in H^1(\mathrm{R}^n)$ as follows.

Theorem 1.4. *Let $g \in H^1(\mathrm{R}^n)$ and $\Omega(x') = \int_0^\infty r^{n-1}g(rx')dr$. Then* (1.8) *holds in the L^p-norm and in the a.e. sense with*

$$c_g = \frac{i\pi^{(n+1)/2}}{\sigma_{n-1}\,\Gamma((n+1)/2)}\sum_{j=1}^{n}\int_{\mathrm{R}^n}(R_j g)(y)\frac{y_j}{|y|}dy, \qquad \sigma_{n-1} = |\Sigma_{n-1}|. \tag{1.10}$$

Theorems 1.3 and 1.4 are multidimensional generalizations of the similar result for $n = 1$ from [11, Sec. 12]. They give a partial answer to problems 1 and 2 mentioned above, and link together Calderón's reproducing formula with singular integrals.

Example 1.5. Let $g(x) = \psi(|x|)\theta(x')$ be such that

$$\theta \in L^1(\Sigma_{n-1}), \quad \int_0^\infty r^{n-1}\psi(r)dr = 0, \quad \int_0^\infty r^{n-1}|\psi(r)\log r|dr < \infty. \tag{1.11}$$

Then the conditions (1.2) are satisfied and $\Omega(x') \equiv \int_0^\infty r^{n-1}g(rx')dx' \equiv 0$. By Theorem 1.3, for $f \in L^p(\mathrm{R}^n)$, $1 < p < \infty$, we have

$$\lim_{\substack{\varepsilon\to 0\\ \rho\to\infty}} \int_\varepsilon^\rho \frac{f * g_t}{t}dt = c_g f, \qquad c_g = \int_0^\infty r^{n-1}\psi(r)\log\frac{1}{r}dr \int_{\Sigma_{n-1}} \theta(x')dx' \ ,$$

in the L^p-norm and in the a.e. sense (the latter holds if $\int_0^{1/2} r^{n-1}|\psi(r)||\log r|^{1+\delta}dr < \infty$ for some $\delta > 0$).

This example generalizes the corresponding results (for radial wavelets) from [12, 13] and shows that *the reproducing formula* (1.1) *also holds for nonradial degenerate wavelet functions* $g(x) = \psi(|x|)\theta(x')$, *satisfying* (1.11) (more generally, one can take any function g satisfying Theorem 1.3 or Theorem 1.4 with $\Omega \equiv 0$).

Example 1.6. Consdier the Calderón-Zygmund singular integral operator (1.7) with $\theta \in H^1(\Sigma_{n-1})$, $\int_{\Sigma_{n-1}} \theta(x')dx' = 0$. Let $\psi(r)$ be a positive integrable decreasing function on $(0,\infty)$, such that $\int_0^\infty \psi(r)dr = 1$. Then $g(x) = |x|^{1-n}\psi(|x|)\theta(x') \in H^1(\mathrm{R}^n)$ (see Lemma 2.6 below), and Theorem 1.4 yields $c_g = 0$, $\Omega(x') = \theta(x')$,

$$T_\theta f = \lim_{\substack{\varepsilon\to 0\\ \rho\to\infty}} \int_\varepsilon^\rho \frac{f * g_t}{t}dt, \qquad f \in L^p(\mathrm{R}^n), \quad 1 < p < \infty, \tag{1.12}$$

in the L^p-norm and a.e. Thus, *each Calderón-Zygmund operator T_θ can be represented in "wavelet form"* (1.12).

The paper is organized as follows. Section 2 contains auxiliary results, mainly related to the Hardy spaces H^1. In Section 3 we prove Theorem 1.1. A new proof of Theorem 1.2 and a proof of Theorems 1.3 and 1.4 are given in Section 4.

Regarding some perspectives, it is our hope that close results can be obtained in different settings for bilinear operators $I(f,g)(x) = \int (W_g f)(x,t)d\nu(t)$ generated by suitable continuous wavelet transforms $(W_g f)(x,t)$, which are integrated with respect to the relevant measure $d\nu(t)$. The case, when g is replaced by a wavelet measure or distribution, also seems to be interesting. One may expect that a suitable choice of g in $I(f,g)$ will lead to certain new singular integrals corresponding to the initial setting.

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2 AUXILIARY RESULTS.

Recall some basic facts related to the Hardy spaces $H^1(\mathrm{R}^n)$ and $H^1(\Sigma_{n-1})$ (for more details see [15], p. 106; [3], p. 591; [4], p. 233).

Definition 2.1. *A function $a(x)$ on R^n supported by a ball $B \subset \mathrm{R}^n$ is called an atom on R^n if*

$$\int_{\mathrm{R}^n} a(x)dx = 0 \ \ and \ \ |a(x)| \leq |B|^{-1}. \tag{2.1}$$

Definition 2.2. *A regular atom on Σ_{n-1} is a function $\alpha(x')$ on Σ_{n-1} supported by a spherical cap $\mathcal{B} \subset \Sigma_{n-1}$ and satisfying the relations*

$$\int_{\Sigma_{n-1}} \alpha(x')dx' = 0, \qquad |\alpha(x')| \leq |\mathcal{B}|^{-1}. \tag{2.2}$$

An exceptional atom on Σ_{n-1} is the constant function $\alpha(x')$, having the value $|\Sigma_{n-1}|^{-1}$.

Theorem 2.3. *An integrable on R^n function g belongs to $H^1(\mathrm{R}^n)$ if and only if*

$$g = \sum_{k=1}^{\infty} \lambda_k a_k \tag{2.3}$$

where a_k is an atom on R^n and $\Sigma_{k=1}^{\infty}|\lambda_k| < \infty$. If $g \in H^1(\mathrm{R}^n)$, then $\|g\|_{H^1} \sim \inf \sum_{k=1}^{\infty}|\lambda_k|$ where the infimum is taken over all decompositions (2.3).

A similar statement holds for functions on Σ_{n-1} provided that atoms in the equality of the (2.3)-type are regular or exceptional (see [3], pp. 591, 592).

We prove now some auxiliary lemmas which are of independent interest.

Lemma 2.4. *If $g \in H^1(\mathrm{R}^n)$, then $\Omega(x') = \int_0^\infty t^{n-1}g(tx')dt \in H^1(\Sigma_{n-1})$, and*

$$\|\Omega\|_{H^1(\Sigma_{n-1})} \leq c\|g\|_{H^1(\mathrm{R}^n)}. \tag{2.4}$$

Proof. Let g be an atom, supported by a ball B of radius δ, $r = dist(0, \bar{B})$. It is clear that $\int_{\Sigma_{n-1}} \Omega(x')dx = \int_{\mathrm{R}^n} g(x)dx = 0$. Let us show that there is a geodesic ball $\mathcal{B} \subset \Sigma_{n-1}$ such that

$$\operatorname{supp}\Omega \subset \mathcal{B} \quad \text{and} \quad |\Omega| \leq \frac{A}{|\mathcal{B}|}, \qquad A = A(n).$$

Put $r = k\delta$ for some $k \geq 0$. Then

$$|\Omega(x')| = \Big| \int\limits_{r}^{r+2\delta} t^{n-1} g(tx') dt \Big| \leq c_n \frac{(r+2\delta)^n - r^n}{\delta^n} = c_n P_{n-1}(k), \tag{2.5}$$

where $P_{n-1}(k) = c_n((k+2)^n - k^n)$ if $k > 0$ and $P_{n-1}(k) = c_n$ if $k = 0$. Let k_0 be a real number such that $P_{n-1}(k)/(k+1)^{n-1} \leq A(n)$ for $k \geq k_0$. Consider two cases: 1) $k \geq k_0$, and 2) $k < k_0$. In the first case, we choose $\mathcal{B}$ to be the projection of B on Σ_{n-1}. If ρ is the radius of $\mathcal{B}$, then $|\mathcal{B}| \sim (\sin\rho)^{n-1} = (\delta/(r+\delta))^{n-1} = (\kappa+1)^{1-n}$. Hence by (2.5),

$$|\Omega(x')| \leq \frac{c_n P_{n-1}(k)}{|\mathcal{B}|(k+1)^{n-1}} \leq \frac{A(n)}{|\mathcal{B}|}.$$

Let $k < k_0$. Then (2.5) shows that one can take $\mathcal{B} = \Sigma_{n-1}$ with $|\Omega(x')| \leq A/|\mathcal{B}|$, $A = c_n P_{n-1}(\kappa_0)$. If $g = \sum_{j=1}^{\infty} \lambda_j a_j$ is an atomic decomposition of g, then $\Omega(g) = \sum_{j=1}^{\infty} \lambda_j \Omega(a_j) = A \sum_{j=1}^{\infty} \lambda_j \tilde{\Omega}(a_j)$, where $\tilde{\Omega}(a_j) = \Omega(a_j)/A$ are regular atoms on Σ_{n-1}, and we are done. Λ

The next statement can be regarded as an inverse of the previous one in a certain sense.

Lemma 2.5. *If $\theta(x') \in H^1(\Sigma_{n-1})$, $\int_{\Sigma_{n-1}} \theta(x')dx' = 0$, and $\varphi(r)$ is an integrable positive decreasing function on* R_+*, then*

$$g(x) \stackrel{\text{def}}{=} \theta(x') \frac{\varphi(|x|)}{|x|^{n-1}} \in H^1(\mathrm{R}^n) \qquad \textit{and} \qquad \|g\|_{H^1(\mathrm{R}^n)} \leq c\, \|\theta\|_{H^1(\Sigma_{n-1})}. \tag{2.6}$$

Remark. This statement was inspired by the argument from [15], p. 178, where $\theta(x')$ was a bounded function.

Proof. Let first $\theta(x')$ be a regular atom, supported by the geodesic ball $\mathcal{B}$ with the center at $\sigma \in \Sigma_{n-1}$ of radius ρ. We claim that $G(x) = \theta(x')\varphi(|x|)|x|^{1-n}$ admits a decomposition $G = \sum_{k=-\infty}^{\infty} c_\kappa G_\kappa$, where G_κ are atoms in $H^1(\mathrm{R}^n)$ and $\sum_{k=-\infty}^{\infty} |c_k| \leq c < \infty$ with c independent of θ. Given a number $a \in (1, 2]$, which will be specified later, we set

$$G_k(x) = \frac{a^{1-k}}{(a-1)\,\varphi(a^k)} \frac{\theta(x')\varphi(|x|)}{|x|^{n-1}} \qquad \text{if} \qquad a^k \leq |x| < a^{k+1},$$

and $G_k(x) \equiv 0$ elsewhere. Then $G = \sum_{k=-\infty}^{\infty} c_k G_k$, where $c_k = a^{k-1}(a-1)\varphi(a^k)$, and

$$\sum_{k=-\infty}^{\infty} c_k = \sum_{k=-\infty}^{\infty} \varphi(a^k)\, a^{k-1}(a-1) < \sum_{k=-\infty}^{\infty} \int\limits_{a^{k-1}}^{a^k} \varphi(r)dr = \int\limits_{0}^{\infty} \varphi(r)dr < \infty. \tag{2.7}$$

Let us show that G_k is an atom for a suitable $a = a(\rho)$. Consider two cases: 1) $\rho \geq \rho_0 = \pi/100$, and 2) $\rho < \rho_0$. In the first case we put $a = 2$ and obtain

$$|G_k| \leq \frac{2^{-k}}{(2^k)^{n-1}|\mathcal{B}|} \leq \frac{c}{|B_k|}, \quad c = c(n, \rho_0), \quad B_k = \{x\colon |x| < 2^{k+1}\}.$$

Since $\operatorname{supp} G_k \subset B_k$ and $\int G_k = 0$, then $c^{-1} G_k$ is an atom. If $\rho < \rho_0$ we proceed as follows. Put $a = \cos\rho/(1 - 2\tan\rho)$. Then there is a cube Q_k around $\mathcal{D}_k \equiv \{x\colon a^k < |x| < a^{k+1},\ x/|x| \in \mathcal{B}\}$ with the axis of symmetry along the vector σ, centered at the point $q_k = 2^{-1}(a^{k+1} + a^k \cos\rho)\sigma$, and having two parallel faces on the hyperplanes $\langle x, \sigma\rangle = a^{k+1}$ and $\langle x, \sigma\rangle = a^k \cos\rho$. A simple calculation shows that the side of Q_k equals $a^k(a - \cos\rho)$. Let $B_k = \{x\colon |x - q_k| < r_k\}$, $r_k = 2^{-1/2} a^k (a - \cos\rho)$, be the smallest ball containing Q_k. Then $\operatorname{supp} G_k \subset B_k$ and

$$|G_k| \leq \frac{a^{1-k}}{(a-1)\, a^{k(n-1)} |\mathcal{B}|} \leq \frac{c_n}{|B_k|} \omega(\rho), \qquad \omega(\rho) = \frac{(a - \cos\rho)^n}{(a-1)|\mathcal{B}|}.$$

Since for small ρ, $|\mathcal{B}| \sim (\sin\rho)^{n-1}$, $a - \cos\rho \sim \sin\rho$ and $a - 1 \sim \sin\rho$, then $\sup\limits_{0<\rho<\rho_0} \omega(\rho) \leq c(n, \rho_0)$ and G_k is an atom (up to a constant multiple independent of k and ρ).

In the general case we have $\theta(x') = \sum_{j=1}^{\infty} \lambda_j \alpha_j(x')$ (see Definition 2.2), where $\alpha_j(x')$ is a regular atom such that $\operatorname{supp} \alpha_j \subset \mathcal{B}_j$, $|\alpha_j| \leq |\mathcal{B}_j|^{-1}$, $\mathcal{B}_j$ being a geodesic ball of radius ρ_j. We set

$$g(x) = \sum_{\rho_j \geq \rho_0} \lambda_j \sum_{k=-\infty}^{\infty} c_{k,j} G_{k,j}(x) + \sum_{\rho_j < \rho_0} \lambda_j \sum_{k=-\infty}^{\infty} c_{k,j} G_{k,j}(x), \quad \rho_0 = \pi/100,$$

with $c_{k,j}$ and $G_{k,j}(x)$ constructed above according to the size of ρ_j. Due to (2.7),

$$\sum_j \sum_k c_{k,j} |\lambda_j| \leq c \sum_j |\lambda_j| \leq c\, \|\theta\|_{H^1(\Sigma_{n-1})},$$

and the proof is complete. Λ

Lemma 2.6. *Let g be an integrable function, such that (a) g satisfies (1.2) or (b) $g \in H^1(\mathbb{R}^n)$. Then $(Kg)(x) = |x|^{-n} \int_{|y|<|x|} g(y)dy \in L^1(\mathbb{R}^n)$ and the constant $c_g \overset{\text{def}}{=} \int_{\mathbb{R}^n} (Kg)(x)dx$ can be evaluated as follows:*

$$c_g = \int\limits_{\mathbb{R}^n} g(x) \log \frac{1}{|x|}\, dx \tag{2.8}$$

in the case (a) and

$$c_g = \frac{i\pi^{(n+1)/2}}{\sigma_{n-1}\, \Gamma((n+1)/2)} \sum_{j=1}^{n} \int\limits_{\mathbb{R}^n} (R_j g)(y) \frac{y_j}{|y|} dy \tag{2.9}$$

in the case (b).

The proof of this lemma can be found in [12, Theorem 3].

Lemma 2.7. *Given a function* $g \in L^1(\mathrm{R}^n)$, *let*

$$k^0(x) = \begin{cases} |x|^{-n} \displaystyle\int_0^{|x|} r^{n-1} g(rx')dr & \text{if } \ |x| < 1, \\ -|x|^{-n} \displaystyle\int_{|x|}^{\infty} r^{n-1} g(rx')dr & \text{if } \ |x| > 1. \end{cases} \tag{2.10}$$

If g *satisfies* (1.2), *then* $k^0 \in L^1(\mathrm{R}^n)$ *and*

$$\int_{\mathrm{R}^n} k^0(x)dx = \int_{\mathrm{R}^n} g(x) \log \frac{1}{|x|}\, dx. \tag{2.11}$$

Moreover, if g *satisfies* (1.9) *and*

$$\|g\|_{\delta,log} = \int_{|x|<1/2} |g(x)|\, |\log|x|\,|^{1+\delta} dx \ + \int_{|x|>1} |g(x)|\, \log|x|\, dx + \|g\|_1, \tag{2.12}$$

then for any $f \in L^p(\mathrm{R}^n)$, $1 < p < \infty$,

$$\| \sup_{\varepsilon>0} |k^0_\varepsilon * f|\, \|_p \le c\, \|g\|_{\delta,log} \|f\|_p, \qquad k^0_\varepsilon(x) = \varepsilon^{-n} k^0(x/\varepsilon). \tag{2.13}$$

Proof. We have

$$\int_{|x|<1} |k^0(x)|dx \le \int_{\Sigma_{n-1}} dx' \int_0^1 \frac{dr}{r} \int_0^r t^{n-1} |g(tx')|dt = \int_{|x|<1} |g(x)|\, \log \frac{1}{|x|} dx,$$

and

$$\int_{|x|>1} |k^0(x)|dx \le \int_{\Sigma_{n-1}} dx' \int_1^\infty \frac{dr}{r} \int_r^\infty t^{n-1} |g(tx')|dt = \int_{|x|>1} |g(x)| \log|x| dx.$$

The proof of (2.11) is similar. Furthermore, since

$$|k^0(x)| \le \frac{1}{|x|^n} \begin{cases} |\log|x||^{-1-\delta} \displaystyle\int_0^{1/2} r^{n-1} |g(rx')| |\log r|^{1+\delta} dr & \text{if } \ |x| \le 1/2, \\ \displaystyle\int_0^1 r^{n-1} |g(rx')|dr & \text{if } \ 1/2 < |x| \le 1, \\ \displaystyle\int_{|x|}^\infty r^{n-1} |g(rx')|dr & \text{if } \ |x| > 1, \end{cases}$$

and $\int_{\mathrm{R}^n} |k^0(x)|dx \le \|g\|_{\delta,\log}$, then (2.13) holds by Proposition 1 from [15], p. 71. Λ

3 PROOF OF THEOREM 1.1.

We keep on the notation (1.4), (1.5), and remind that $g \in L^1(\mathrm{R}^n)$, $f \in L^p(\mathrm{R}^n)$, $1 < p < \infty$, $I_{\varepsilon,\rho}(f,g)(x) = \int_\varepsilon^\rho (f * g_t)(x)dt/t$, $0 < \varepsilon < \rho < \infty$. The required result will be derived from the following two lemmas.

Lemma 3.1. *If g is odd, then*

$$\| \sup_{0<\varepsilon<\rho<\infty} |I_{\varepsilon,\rho}(f,g)| \|_p \leq c \, \|g\|_1 \|f\|_p, \tag{3.1}$$

and

$$\lim_{\substack{\varepsilon \to 0 \\ \rho \to \infty}} I_{\varepsilon,\rho}(f,g)(x) = \frac{\pi i}{2} \int_{\mathrm{R}^n} g(y)(H_{y'} f)(x) dy \tag{3.2}$$

in the L^p-norm and almost everywhere.

Proof. By changing the order of integration and taking into account that g is odd, we have

$$I_{\varepsilon,\rho}(f,g)(x) = \frac{1}{2} \int_{\mathrm{R}^n} g(y) dy \int_{\varepsilon|y|<|t|<\rho|y|} \frac{f(x - ty')}{t} \, dt.$$

It remains to apply an argument which is similar to that in [16], Ch. VI, Sec. 2. Λ

Lemma 3.2. *If $g \in H^1(\mathrm{R}^n)$ is even, then* (3.1) *holds with $\|g\|_1$ replaced by $\|g\|_{H^1}$ and*

$$\lim_{\substack{\varepsilon \to 0 \\ \rho \to \infty}} I_{\varepsilon,\rho}(f,g)(x) = \frac{\pi i}{2} \sum_{j=1}^{n} \int_{\mathrm{R}^n} (R_j g)(y)(H_{y'} R_j f)(x) dy \tag{3.3}$$

in the L^p-norm and almost everywhere.

Proof. Following the Calderón-Zygmund idea, we pass from the "even case" to the "odd one" by employing the Riesz transforms R_j. Since $\|R_j f\|_p \leq c\|f\|_p$, $\|R_j g\|_{H^1} \leq c\|g\|_{H^1}$, and $R_j g$ is odd, then (3.1) (with $\|g\|_{H^1}$) and (3.3) are consequences of the equality $I_{\varepsilon,\rho}(f,g) = \sum_{j=1}^n I_{\varepsilon,\rho}(R_j f, R_j g)$ (use Lemma 3.1). The latter can be easily checked for Schwartz functions f and g orthogonal to all polynomials, by applying the Fourier transform to both sides and using the equalities

$$(R_j f)^\wedge(\xi) = \hat{f}(\xi) \frac{\xi_j}{|\xi|}, \quad j = 1, ..., n, \qquad \sum_{j=1}^{n} \frac{\xi_j^2}{|\xi|^2} = 1.$$

We recall that the space $\Phi(\mathrm{R}^n)$ of all such functions is dense in $L^p(\mathrm{R}^n)$ and in $H^1(\mathrm{R}^n)$ ([15], p. 128). Λ

In order to prove Theorem 1.1 we write g in the form $g = g_+ + g_-$, $g_\pm(x) = (g(x) + g(-x))/2$. The result then follows by Lemmas 3.1 and 3.2.

4 PROOF OF THEOREMS 1.2, 1.3 AND 1.4.

The following lemma will be used repeatedly as a bridge between Theorems 1.1 - 1.3. We assume that $f \in L^p(\mathrm{R}^n)$, $1 < p < \infty$, $g \in L^1(\mathrm{R}^n)$ satisfies (1.2), and keep on the notation $I_{\varepsilon,\rho}(f,g)$ and $T_\Omega^{\varepsilon,\rho} f$ from Theorem 1.3.

Lemma 4.1. *(i) For $\varepsilon \to 0$ and $\rho \to \infty$, the integrals $I_{\varepsilon,\rho}(f,g)$ and $T_\Omega^{\varepsilon,\rho} f$ converge in the L^p-norm simultaneously. If the limit $T_\Omega f = \lim^{(L^p)}_{\substack{\varepsilon\to 0\\ \rho\to\infty}} T_\Omega^{\varepsilon,\rho} f$ exists, then*

$$\lim^{(L^p)}_{\substack{\varepsilon\to 0\\ \rho\to\infty}} I_{\varepsilon,\rho}(f,g) = c_g f + T_\Omega f, \quad c_g = \int_{\mathrm{R}^n} g(x) \log \frac{1}{|x|} dx. \tag{4.1}$$

(ii) Under the additional assumption (1.9) the following assertions hold :

$$(a) \qquad \| \sup_{0<\varepsilon<\rho<\infty} |T_\Omega^{\varepsilon,\rho} f| \|_p \le c_1 \, (\|g\|_{\delta,log} + \|g\|_{H^1}) \|f\|_p \tag{4.2}$$

if and only if

$$\| \sup_{0<\varepsilon<\rho<\infty} |I_{\varepsilon,\rho}(f,g)| \|_p \le c_2 \, (\|g\|_{\delta,log} + \|g\|_{H^1}) \|f\|_p \tag{4.3}$$

where c_1, c_2 are some constants independent of f and g, and $\|g\|_{\delta,log}$ is defined by (2.12).

(b) The statements in (i) are valid with the L^p-convergence replaced by that in the a.e. sense.

Proof. (i) A simple calculation yields

$$I_{\varepsilon,\rho}(f,g) = f * k_{\varepsilon,\rho}, \quad \text{where} \quad k_{\varepsilon,\rho}(x) = |x|^{-n} \int_{|x|/\rho}^{|x|/\varepsilon} r^{n-1} g(rx') dr \ (\in L^1(\mathrm{R}^n)). \tag{4.4}$$

Let $k(x) = |x|^{-n} \int_0^{|x|} r^{n-1} g(rx') dr$. One can write $k(x) = k^0(x) + q^0(x)$, where $k^0(x)$ is the function (2.10), $q^0(x) \equiv 0$ for $|x| < 1$ and $q^0(x) = |x|^{-n}\Omega(x')$ for $|x| > 1$. Then (4.4) reads:

$$I_{\varepsilon,\rho}(f,g) = k^0_\varepsilon * f - k^0_\rho * f + T_\Omega^{\varepsilon,\rho} f \tag{4.5}$$

(the subscripts indicate the relevant dilations, e.g., $k^0_\varepsilon(x) = \varepsilon^{-n} k^0(x/\varepsilon)$), and the results follow by Lemma 2.7. Λ

Proof of Theorem 1.2. Let us prove the maximal estimate

$$\| \sup_{0<\varepsilon<\rho<\infty} |T_\theta^{\varepsilon,\rho} f| \, \|_p \le c \, \|\theta\|_{H^1(\Sigma_{n-1})} \|f\|_p. \tag{4.6}$$

We set $g(x)=e^{-|x|}|x|^{1-n}\theta(x')$. By Lemma 2.5, $g\in H^1(\mathrm{R}^n)$ and $\|g\|_{H^1(\mathrm{R}^n)}\le c\|\theta\|_{H^1(\Sigma_{n-1})}$. Further, Theorem 1.1 yields $\|\sup\limits_{0<\varepsilon<\rho<\infty}|I_{\varepsilon,\rho}(f,g)|\,\|_p\le c\|g\|_{H^1(\mathrm{R}^n)}\|f\|_p$. By taking all of these into account, we make use of Lemma 4.1 (ii a). In our case $\Omega(x')=\theta(x')$, and

$$\|g\|_{\delta,\log}=\|\theta\|_1\Big(\int_0^{1/2}|\log r|^{1+\delta}e^{-r}dr+\int_1^\infty|\log r|\,e^{-r}dr+1\Big)\le c\,\|\theta\|_{H^1(\Sigma_{n-1})}.$$

Hence (4.6) follows. The estimate (4.6) enables us to obtain an a.e. convergence of $T_\theta^{\varepsilon,\rho}f$ by taking into account that for compactly supported smooth f,

$$(T_\theta^{\varepsilon,\rho}f)(x)\equiv\int\limits_{\varepsilon<|y|<1}(f(x-y)-f(x))\frac{\theta(y')}{|y|^n}dy+\int\limits_{1<|y|<\rho}f(x-y)\frac{\theta(y')}{|y|^n}dy$$

converges pointwise as $\varepsilon\to 0$ and $\rho\to\infty$. The L^p-convergence of $T_\theta^{\varepsilon,\rho}f,\ f\in L^p(\mathrm{R}^n)$, then follows by the Lebesgue dominated convergence theorem. Λ

Remark 4.2. Variants of Theorem 1.2 are formulated sometimes in terms of the truncated integral $(T_\theta^\varepsilon f)(x)=\int_{|y|>\varepsilon}f(x-y)\theta(y')dy/|y|^n$ (without truncation at infinity). For $f\in L^p(\mathrm{R}^n)$, $1\le p<\infty$, and each $\theta\in L^1(\Sigma_{n-1})$ such an integral is well-defined a.e. for $\varepsilon>0$ [1, p. 292]. Since

$$\|\sup_{\varepsilon>0}|T_\theta^\varepsilon f|\|_p=\|\sup_{\varepsilon>0}|\lim_{\rho\to\infty}T_\theta^{\varepsilon,\rho}f|\|_p\le\|\sup_{0<\varepsilon<\rho<\infty}|T_\theta^{\varepsilon,\rho}f|\|_p,$$

then Theorem 1.2 can be reformulated in terms of $T_\theta^\varepsilon f$.

Proof of Theorem 1.3. By Theorem 1.2, $T_\Omega f=\lim\limits_{\substack{\varepsilon\to 0\\ \rho\to\infty}}T_\Omega^{\varepsilon,\rho}f$ exists in the L^p-norm and a.e., and $\|\sup\limits_{0<\varepsilon<\rho<\infty}|T_\Omega^{\varepsilon,\rho}f|\,\|_p\le c\,\|\Omega\|_{H^1(\Sigma_{n-1})}\|f\|_p$. Hence the first statement of the theorem holds by Lemma 4.1 (i). According to (4.5),

$$\sup_{0<\varepsilon<\rho<\infty}|I_{\varepsilon,\rho}(f,g)|\le 2\sup_{\varepsilon>0}|k_\varepsilon^0*f|+\sup_{0<\varepsilon<\rho<\infty}|T_\Omega^{\varepsilon,\rho}f|,$$

and the second statement follows by Lemma 2.7. Λ

Proof of Theorem 1.4. We first prove the part of the theorem, related to the L^p-convergence. Let $g\in\Phi(\mathrm{R}^n)$ (the space of Schwartz functions orthogonal to all polynomials). We recall that $\Phi(\mathrm{R}^n)$ is dense in $L^p(\mathrm{R}^n)$ and in $H^1(\mathrm{R}^n)$ ([15], p. 128). Then (1.8) gives

$$I(f,g)=c_gf+T_\Omega f,\quad f\in L^p(\mathrm{R}^n),\quad g\in\Phi(\mathrm{R}^n),\tag{4.7}$$

where, by Lemma 2.6,

$$c_g=\int_{\mathrm{R}^n}(Kg)(x)dx=\frac{i\pi^{(n+1)/2}}{\sigma_{n-1}\Gamma((n+1)/2)}\sum_{j=1}^n\int\limits_{\mathrm{R}^n}(R_jg)(y)\frac{y_j}{|y|}dy.$$

Fix $f \in L^p(\mathrm{R}^n)$ and consider the mappings $g \to I(f,g)$, $g \to c_g f$, $g \to T_\Omega f$ as operators from $H^1(\mathrm{R}^n)$ to $L^p(\mathrm{R}^n)$ (the first and the third operators are defined as L^p-limits on $g \in \Phi(\mathrm{R}^n)$ for fixed $f \in L^p(\mathrm{R}^n)$). By Theorem 1.1, $||I(f,g)||_p \leq ||g||_{H^1(\mathrm{R}^n)}||f||_p$. Note also that $||c_g f||_p \leq ||g||_{H^1(\mathrm{R}^n)}||f||_p$ (due to the boundedness of $R_j : H^1(\mathrm{R}^n) \to L^1(\mathrm{R}^n)$). Furthermore, by Theorem 1.2 and Lemma 2.4,

$$||T_\Omega f||_p \leq c\,||\Omega||_{H^1(\Sigma_{n-1})}||f||_p \leq c\,||g||_{H^1(\mathrm{R}^n)}||f||_p.$$

Owing to these relations, one can extend (4.7) to all $g \in H^1(\mathrm{R}^n)$. It remains to show that our extensions, say, $\tilde{I}(f,g)$ and $\tilde{T}_\Omega f$, coincide with

$$I(f,g) = \overset{(L^p)}{\lim_{\substack{\varepsilon\to 0\\ \rho\to\infty}}} I_{\varepsilon,\rho}(f,g) \quad \text{and} \quad T_\Omega f = \overset{(L^p)}{\lim_{\substack{\varepsilon\to 0\\ \rho\to\infty}}} T_\Omega^{\varepsilon,\rho} f$$

respectively for an arbitrary $g \in H^1(\mathrm{R}^n)$. Given $g \in H^1(\mathrm{R}^n)$, let $\{g_k\}_{k=1}^\infty \subset \Phi(\mathrm{R}^n)$ be a sequence such that $||g - g_k||_{H^1(\mathrm{R}^n)} \to 0$ as $k \to \infty$. Then

$$||\tilde{I}(f,g) - I(f,g)||_p \leq \sum_{j=1}^{4} \Delta_j \quad \text{where} \quad \Delta_1 = ||\tilde{I}(f,g) - I(f,g_k)||_p,$$

$$\Delta_2 = ||I(f,g_k) - I_{\varepsilon,\rho}(f,g_k)||_p, \qquad \Delta_3 = ||I_{\varepsilon,\rho}(f,g_k - g)||_p,$$

$$\Delta_4 = ||I_{\varepsilon,\rho}(f,g) - I(f,g)||_p.$$

For sufficiently large k, Δ_1 becomes arbitrary small by definition of $\tilde{I}(f,g)$. The same holds for Δ_3 due to (1.6). The quantities Δ_2 and Δ_4 tend to 0 as $\varepsilon \to 0$ and $\rho \to \infty$. Hence $\tilde{I}(f,g) = I(f,g)$. Similarly, $\tilde{T}_\Omega f = T_\Omega f$. Thus, we have (4.7) with $I(f,g)$ and $T_\Omega f$ defined as the L^p-limits. The existence of these limits in the a.e. sense follows by Theorems 1.1 and 1.2. Λ

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