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The unit ball is an attractor of the intersection body operator ☆

Alexander Fish [a], Fedor Nazarov [a], Dmitry Ryabogin [b,*], Artem Zvavitch [b]

[a] *Department of Mathematics, University of Wisconsin–Madison, 480 Lincoln Drive, Madison, WI 53706, United States*
[b] *Department of Mathematics, Kent State University, Kent, OH 44242, United States*

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Abstract

The intersection body of a ball is again a ball. So, the unit ball $B_d \subset \mathbb{R}^d$ is a fixed point of the intersection body operator acting on the space of all star-shaped origin symmetric bodies endowed with the Banach–Mazur distance. E. Lutwak asked if there is any other star-shaped body that satisfies this property. We show that this fixed point is a local attractor, i.e., that the iterations of the intersection body operator applied to any star-shaped origin symmetric body sufficiently close to B_d in Banach–Mazur distance converge to B_d in Banach–Mazur distance. In particular, it follows that the intersection body operator has no other fixed or periodic points in a small neighborhood of B_d. We will also discuss a harmonic analysis version of this question, which studies the Radon transforms of powers of a given function.
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* Corresponding author.
E-mail addresses: afish@math.wisc.edu (A. Fish), nazarov@math.wisc.edu (F. Nazarov), ryabogin@math.kent.edu (D. Ryabogin), zvavitch@math.kent.edu (A. Zvavitch).

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1. Introduction

The notion of an *intersection body of a star body* was introduced by E. Lutwak [7]: K is called the intersection body of L if the radial function of K in every direction is equal to the $(d-1)$-dimensional volume of the central hyperplane section of L perpendicular to this direction:

$$\rho_K(\xi) = \mathrm{vol}_{d-1}\big(L \cap \xi^{\perp}\big), \quad \forall \xi \in S^{d-1}, \tag{1.1}$$

where $\rho_K(\xi) = \sup\{a\colon a\xi \in K\}$ is the radial function of the body K and $\xi^{\perp} = \{x \in \mathbb{R}^d\colon (x,\xi) = 0\}$ is the central hyperplane perpendicular to the vector ξ. Using the formula for the volume in polar coordinates in $\xi^{\perp}$, we derive the following analytic definition of an *intersection body of a star body*: K is the intersection body of L if

$$\rho_K(\xi) = \frac{1}{d-1}\mathcal{R}\rho_L^{d-1}(\xi) := \frac{1}{d-1}\int\limits_{S^{d-1}\cap\xi^{\perp}} \rho_L^{d-1}(\theta)\,d\theta.$$

Here $\mathcal{R}$ stands for the spherical Radon transform. We refer the reader to books [2, Section 8, p. 304], [6, Chapter 4, p. 71] for more information on the definition and properties of intersection bodies of star bodies and the role they play in Convex Geometry and Geometric Tomography.

Let us denote by $\mathcal{I}L$ the intersection body of a body L. Let $\mathbb{S}_d$ be the set of all equivalence classes of star-shaped origin symmetric bodies in $\mathbb{R}^d$ (two bodies are equivalent if they can be obtained from each other by a linear transformation). We endow $\mathbb{S}_d$ with the Banach–Mazur distance

$$d_{BM}(K,L) = \inf\big\{b/a\colon \exists T \in GL(d) \text{ such that } aK \subseteq TL \subseteq bK\big\}.$$

We note that $\mathcal{I}(TL) = |\det T|(T^*)^{-1}(\mathcal{I}L)$, for all $T \in GL(d)$ (see [2, Theorem 8.1.6]), hence the action of $\mathcal{I}$ on $\mathbb{S}_d$ is well defined, and $d_{BM}(\mathcal{I}(TK), \mathcal{I}(TL)) = d_{BM}(\mathcal{I}K, \mathcal{I}L)$.

The action of $\mathcal{I}$ on $\mathbb{S}_2$ is quite simple; since $\mathcal{I}L$ is just L rotated by $\pi/2$ and stretched 2 times, we have $\mathcal{I}L = L$ in $\mathbb{S}_2$, so every point of $\mathbb{S}_2$ is a fixed point of $\mathcal{I}$.

Let B_d be the unit Euclidean ball. We have

$$\rho_{\mathcal{I}(B_d)}(\xi) = \mathrm{vol}_{d-1}\big(B_d \cap \xi^{\perp}\big) = \mathrm{vol}_{d-1}(B_{d-1}).$$

Thus, B_d is a fixed point of $\mathcal{I}$ in $\mathbb{S}_d$.

Question. Do there exist any other fixed or periodic points of $\mathcal{I}$ in $\mathbb{S}_d$, $d \geqslant 3$?

In this paper we show that there are no such points in a small neighborhood of the ball B_d. This will immediately follow from the following

Theorem 1.

$$\mathcal{I}^m L \xrightarrow{\mathbb{S}_d} B_d \quad \textit{as } m \to \infty,$$

for all L sufficiently close to B_d in the Banach–Mazur distance.

Corollary. *If $\mathcal{I}^m L = L$ for some $m \geqslant 1$ and L is close to B_d in Banach–Mazur distance, then L is an ellipsoid.*

More information on this and analogous questions can be found in Chapter 8 of [2] (see Problems 8.6 and 8.7, p. 337 and Note 8.6, p. 341) and [8,3].

We also note that a similar question for projection bodies (see [2,6]) is much better understood. It is quite easy to observe that the projection body of a cube is again (a dilation of) a cube. W. Weil (see [12]) described the polytopes that are stable under the projection body operation. Still the general question of the description of all fixed points remains open.

Notation. For a convex body $K \subset \mathbb{R}^d$, consider the following two quantities:

$$d_\infty(K) = \inf\{\|1 - \rho_{TK}\|_\infty\colon\ T \in GL(d)\},$$
$$d_2(K) = \inf\{\|1 - \rho_{TK}\|_2\colon\ T \in GL(d)\}.$$

Note that in the small neighborhood of B_d, the ratio $d_\infty(K)/\log d_{BM}(K, B_d)$ is bounded from both above and below by positive constants.

In this paper, we will denote by $|u|$ the Euclidean norm of a vector $u \in \mathbb{R}^d$. We will denote by C, c constants depending on d (dimension) only, which may change from line to line.

2. Plan of the proof of Theorem 1

To avoid writing irrelevant normalization constants in formulae, from now on, we shall denote by $\mathcal{R}$ the normalized Radon transform on S^{d-1} that differs from the usual one by the factor $\frac{1}{\mathrm{vol}_{d-2}(S^{d-2})}$, so $\mathcal{R}1 = 1$. It doesn't change anything because homotheties have already been factored out in the definition of $\mathbb{S}_d$.

Our main tool in the proof of Theorem 1 is spherical harmonics. We refer the reader to [4] for more information and definitions. We denote by $\mathcal{H}_k$ the space of spherical harmonics of degree k. We shall denote by H_k^f the projection of f to $\mathcal{H}_k$, so f is represented by the series $\sum_{k\geqslant 0} H_k^f$.

The following formula for the Radon transform of a spherical harmonic $H_k \in \mathcal{H}_k$ of even order k is especially useful for our calculations (see [4, Lemma 3.4.7]):

$$\mathcal{R}H_k = (-1)^{k/2} v_{d,k} H_k, \tag{2.1}$$

where

$$v_{d,k} = \frac{1 \cdot 3 \cdot \ldots \cdot (k-1)}{(d-1)(d+1)\ldots(d+k-3)} \approx k^{-(d-2)}.$$

Let $K \in \mathbb{S}_d$ be close to B_d. Our main goal is to show the following two things:

(1) $\mathcal{I}^m K$ is smooth for all large m.
(2) If K is sufficiently smooth and close to B_d, then $d_2(\mathcal{I}K) \leqslant \lambda d_2(K)$ with some $\lambda < 1$.

The first claim will follow from the smoothing properties of $\mathcal{R}$. Since $f : S^{d-1} \to \mathbb{R}$ is C^m-smooth essentially if the norms of H_k^f decay as k^{-m} and since $\mathcal{R}f \sim \sum_{k\geqslant 0}(-1)^{k/2} v_{d,k} H_k^f$, we conclude that the order of smoothness of $\mathcal{R}f$ exceeds the order of smoothness of f by roughly speaking $d-2 \geqslant 1$.

Raising f to the power $d-1$ does not change its smoothness class but can drastically increase the norm of f in that class, so we shall need some accurate computation to show that the smoothing effect still prevails if f is close to constant.

To prove the second claim, we write $\rho_K = 1+\varphi$, where φ is an even function with small L^∞-norm and $\int_{S^{n-1}} \varphi = 0$. Then

$$\begin{aligned}\rho_{\mathcal{I}K} &= 1+(d-1)\mathcal{R}\varphi + \mathcal{R}\sum_{i=2}^{d-1}\binom{d-1}{i}\varphi^i \\ &= 1+(d-1)\mathcal{R}\varphi + \mathcal{R}O\big(\varphi^2\big).\end{aligned}$$

The main idea is to try to show that $\|(d-1)\mathcal{R}\varphi\|_{L^2} \leqslant \lambda\|\varphi\|_{L^2}$ with some $\lambda<1$. Since $\|\varphi^2\|_{L^2} = O(\|\varphi\|_{L^\infty}\|\varphi\|_{L^2})$, and $\|\mathcal{R}\|_{L^2\to L^2} \leqslant 1$, we get $\|\mathcal{R}O(\varphi^2)\|_{L^2} \leqslant C\|\varphi\|_{L^\infty}\|\varphi\|_{L^2}$. Thus,

$$\big\|\mathcal{R}O\big(\varphi^2\big)\big\|_{L^2} \leqslant \frac{1-\lambda}{2}\|\varphi\|_{L^2}, \quad \text{provided that } \|\varphi\|_{L^\infty} \leqslant \frac{1-\lambda}{2},$$

so the last term won't give us any trouble.

Note that $\varphi \sim \sum_{l\geqslant 1} H_{2l}^\varphi$ and the terms H_{2l}^φ are orthogonal. If all the products $v_{d,2l}(d-1)$ were less than 1, our task would be trivial. Unfortunately, $v_{d,2}(d-1)=1$ (but $v_{d,2l}(d-1) \leqslant \frac{3}{d+1} \leqslant \frac{3}{4}$, for $l>1$). Thus, we need to kill H_2^φ somehow. It turns out that it can be done by first applying a suitable linear transformation to K.

Remark 1. The proof below can be noticeably shortened in the case of convex bodies. Then we may use the Busemann theorem (see [1] or [9, Theorem 3.9]; [2, Theorem 8.10]) to claim that $\mathcal{I}^m L$ is convex, for all $m \geqslant 1$, and compare L^∞ and L^2 norms of radial functions of convex bodies directly, avoiding the smoothening procedure.

3. Auxiliary lemmata

For a function $f : S^{d-1} \to \mathbb{R}$ we define its homogeneous extension $\check{f}$ of degree 0 by

$$\check{f}(x) = f\left(\frac{x}{|x|}\right),$$

so if f is a smooth function on S^{d-1}, then $\check{f}$ is a smooth function on $\mathbb{R}^d \setminus \{0\}$. By Df and $D^2 f$, we mean the restrictions to the unit sphere S^{d-1} of the first and the second differentials of $\check{f}$. Note that $D\check{f}$ and $D^2\check{f}$ are homogeneous functions on $\mathbb{R}^d \setminus \{0\}$ of degree -1 and -2 respectively, so the norms $\|Df\|_{L^\infty}$ and $\|D^2 f\|_{L^\infty}$ do not bound the differentials $D\check{f}$ and $D^2\check{f}$ on the entire space $\mathbb{R}^d \setminus \{0\}$. Here $\|Df\|_\infty = \sup_{x\in S^{d-1}} \|D_x f\|$, where $\|T\|$ stands for the usual operator norm of T, and similarly for $D^2 f$. Still they bound them (up to a constant factor) outside any ball of positive radius centered at the origin, which is enough to transfer to the sphere all usual estimates coming from the second order Taylor formula in $\mathbb{R}^d$.

Lemma 1. *Suppose that* $f : S^{d-1} \to \mathbb{R}$ *satisfies* $\|D^2 f\|_{L^\infty} \leqslant 1$, $\|f\|_{L^2} < \varepsilon$, *for some* $\varepsilon \in (0, 1)$. *Then* $\|f\|_{L^\infty} \leqslant C\varepsilon^{\frac{4}{d+3}}$ *and* $\|Df\|_{L^\infty} \leqslant C\varepsilon^{\frac{2}{d+3}}$, *for some* $C = C(d) > 0$.

Proof. Replacing f by $-f$, if necessary, we may assume that

$$\|f\|_{L^\infty} = \max_{S^{d-1}} f = f(x_0) = M > 0.$$

Since $D_{x_0} f = 0$, we can use the second order Taylor formula to conclude that

$$f(x) \geqslant M - C\left\|D^2 f\right\|_{L^\infty}|x - x_0|^2 \geqslant M - C|x - x_0|^2.$$

Thus, in the ball of radius $c\sqrt{M}$ (if M is very large then this ball is just S^{d-1}), centered at x_0, we have

$$f(x) \geqslant M - Cc^2 M \geqslant \frac{1}{2}M, \quad \text{provided that } c^2 C < \frac{1}{2}.$$

Hence,

$$\varepsilon^2 \geqslant \int\limits_{S^{d-1}} f^2 \geqslant c' \frac{M^2}{4}(\sqrt{M})^{d-1} = c' M^{\frac{d+3}{2}}$$

if $c\sqrt{M} < 1$, or

$$\varepsilon^2 \geqslant \frac{M^2}{4},$$

if $c\sqrt{M} \geqslant 1$. In both cases the first inequality follows immediately.

The second inequality can now easily be derived from the classical Landau–Kolmogorov inequality (see [5])

$$\|Df\|_{L^\infty} \leqslant C\|f\|_{L^\infty}^{\frac{1}{2}} \left\|D^2 f\right\|_{L^\infty}^{\frac{1}{2}}. \qquad \square$$

Let $T \in GL(d)$. We would like to define the action of T on bounded functions on S^{d-1} in such a way that, for the radial function $\rho_K(x) = \|x\|_K^{-1}$ of a star-shaped body K, the image $T\rho_K$ would coincide with the radial function of $T^{-1}K$. To this end, note that

$$\rho_{T^{-1}K}(x) = \|Tx\|_K^{-1} = \left\|\frac{Tx}{|Tx|}\right\|_K^{-1} |Tx|^{-1} = \rho_K\left(\frac{Tx}{|Tx|}\right)|Tx|^{-1}.$$

Thus for an arbitrary bounded function $f : S^{d-1} \to \mathbb{R}$, it is natural to put

$$Tf(x) := f\big(\omega_T(x)\big)|Tx|^{-1}, \tag{3.1}$$

where $\omega_T : S^{d-1} \to S^{d-1}$ is given by $\omega_T(x) = \frac{Tx}{|Tx|}$.

Lemma 2. *Let $T = I + Q$, where Q is self-adjoint and $\|Q\| < \frac{1}{2}$. Then*

$$\left|\omega_T(x) - x\right| \leqslant C\|Q\| \quad \textit{for all } x \in S^{d-1}.$$

Proof.

$$\begin{aligned}\left|\omega_T(x) - x\right| &= \frac{1}{|Tx|}\left|Tx - |Tx|x\right| \leqslant \left\|T^{-1}\right\|\left|(Tx - x) - \left(|Tx| - 1\right)x\right| \\ &\leqslant \left\|T^{-1}\right\|\left[|Tx - x| + \left||Tx| - 1\right|\right] \leqslant 2\left\|T^{-1}\right\|\|Q\| \leqslant \frac{2}{1 - \|Q\|}\|Q\|. \qquad \square\end{aligned}$$

4. Classes $\mathcal{U}_\alpha$

Let $\alpha \geqslant 0$. For a bounded function f on S^{d-1}, define $\|f\|_{\mathcal{U}_\alpha}$ to be the least constant M such that $\|f\|_{L^\infty} \leqslant M$ and for every $n \geqslant 1$, there exists a polynomial p_n of degree n satisfying $\|f - p_n\|_{L^2} \leqslant Mn^{-\alpha}$. We will say that $f \in \mathcal{U}_\alpha$ if $\|f\|_{\mathcal{U}_\alpha} < \infty$.

Fix an infinitely smooth function Θ on $[0, +\infty)$ such that $\Theta = 1$ on $[0, 1]$, $\Theta = 0$ on $[2, +\infty)$, and $0 \leqslant \Theta \leqslant 1$ everywhere.

Consider the multiplier operator

$$\mathcal{M}_n f = \mathcal{M}_n^\Theta f = \sum_{k \geqslant 0} \Theta\left(\frac{k}{n}\right) H_k^f. \tag{4.1}$$

We will use the following property: $\|\mathcal{M}_n\|_{L^p \to L^p} \leqslant C(\Theta)$ for all $1 \leqslant p \leqslant \infty$. This result is well known to experts but, for the sake of completeness, we will present a proof in Appendix A.

Note that $\mathcal{M}_n f$ is a polynomial of degree $2n$. Also $\mathcal{M}_n p_n = p_n$ for all polynomials p_n of degree n.

Suppose now that $f \in \mathcal{U}_\alpha$. Let $q_n = \mathcal{M}_n f$. We have

$$\|f - q_n\|_{L^2} = \left\|(f - p_n) - \mathcal{M}_n(f - p_n)\right\|_{L^2} \leqslant C\|f - p_n\|_{L^2} \leqslant C\|f\|_{\mathcal{U}_\alpha} n^{-\alpha},$$

and

$$\|q_n\|_{L^\infty} \leqslant C\|f\|_{L^\infty} \leqslant C\|f\|_{\mathcal{U}_\alpha}.$$

Now we use the polynomials q_n to prove the following lemma describing the properties of the classes $\mathcal{U}_\alpha$.

Lemma 3.

(1) *If $f, g \in \mathcal{U}_\alpha$, then $fg \in \mathcal{U}_\alpha$ and $\|fg\|_{\mathcal{U}_\alpha} \leqslant C\|f\|_{\mathcal{U}_\alpha}\|g\|_{\mathcal{U}_\alpha}$.*
(2) *Let $T \in GL(d)$ with $\|T\|$, $\|T^{-1}\| \leqslant 2$. Then, for every $\delta > 0$, $f \in \mathcal{U}_\alpha$, we have $Tf \in \mathcal{U}_{\alpha-\delta}$ and $\|Tf\|_{\mathcal{U}_{\alpha-\delta}} \leqslant C_\delta\|f\|_{\mathcal{U}_\alpha}$.*
(3) *If $f \in \mathcal{U}_\alpha$, then $\mathcal{R}f \in \mathcal{U}_{\alpha+d-2}$ and $\|\mathcal{R}f\|_{\mathcal{U}_{\alpha+d-2}} \leqslant C\|f\|_{\mathcal{U}_\alpha}$.*

Proof. (1) We obviously have

$$\|fg\|_{L^\infty} \leqslant \|f\|_{L^\infty}\|g\|_{L^\infty} \leqslant \|f\|_{U_\alpha}\|g\|_{U_\alpha}.$$

Now notice that

$$\|f - \mathcal{M}_n f\|_{L^2} \leqslant C\|f\|_{U_\alpha} n^{-\alpha} \quad \text{and} \quad \|g - \mathcal{M}_n g\|_{L^2} \leqslant C\|g\|_{U_\alpha} n^{-\alpha},$$
$$\|\mathcal{M}_n f\|_{L^\infty} \leqslant C\|f\|_{L^\infty} \leqslant C\|f\|_{U_\alpha},$$
$$\|\mathcal{M}_n g\|_{L^\infty} \leqslant C\|g\|_{L^\infty} \leqslant C\|g\|_{U_\alpha},$$

and that $p_n = \mathcal{M}_n f \cdot M_n g$ is a polynomial of degree $4n$. Hence

$$\begin{aligned}\|fg - p_n\|_{L^2} &= \big\|(f - \mathcal{M}_n f)g + M_n f(g - M_n g)\big\|_{L^2} \\ &\leqslant \big\|(f - \mathcal{M}_n f)\big\|_{L_2}\|g\|_{L^\infty} + \|M_n f\|_{L^\infty}\big\|(g - M_n g)\big\|_{L^2} \\ &\leqslant C\|f\|_{U_\alpha}\|g\|_{U_\alpha} n^{-\alpha}.\end{aligned}$$

(2) Write $f = p_n + g$ where $p_n = \mathcal{M}_n f$ and $\|g\|_{L^2} \leqslant C\|f\|_{U_\alpha} n^{-\alpha}$. We have

$$(Tf)(x) = |Tx|^{-1} f\big(\omega_T(x)\big) = |Tx|^{-1} p_n\big(\omega_T(x)\big) + |Tx|^{-1} g\big(\omega_T(x)\big).$$

Since $|Tx|^{-1} \leqslant \|T^{-1}\| \leqslant 2$ on S^{d-1} and ω_T is a diffeomorphism of the unit sphere with bounded volume distortion coefficient, the L^2-norm of the second term does not exceed $C\|g\|_{L^2} \leqslant C\|f\|_{U_\alpha} n^{-\alpha}$. Note now that $x \to |Tx|^{-1}$ is a C^∞-function and ω_T is a C^∞-mapping on S^{d-1}. Moreover, their derivatives of all orders are bounded by some constants depending on the dimension and the order, but not on T (as long as $\|T\|$, $\|T^{-1}\| \leqslant 2$).

We need the following approximation lemma (see for example [10, Theorem 3.3]):

Lemma 4. *If $m \in \mathbb{N}$, $h \in C^m(S^{d-1})$, then for every N, there exists a polynomial P_N of degree N such that $\|h - P_N\|_{L^2} \leqslant C_m\|h\|_{C^m} N^{-m}$.*

Since both the multiplication by a C^∞-function and a C^∞ change of variable are bounded operators in C^m, the function $h(x) = |Tx|^{-1} p_n(\omega_T(x))$ belongs to C^m and $\|h\|_{C^m} \leqslant C_m\|p_n\|_{C^m}$. By the Bernstein inequality (see [11, Theorem 3.2.6]),

$$\|p_n\|_{C^m} \leqslant C_m\|p_n\|_{L^\infty} n^m \leqslant C_m\|f\|_{L^\infty} n^m \leqslant C_m\|f\|_{U_\alpha} n^m.$$

Thus we can find a polynomial P_N of degree $N^{1+\varepsilon}$ such that

$$\|h - P_N\|_{L^2} \leqslant C_m\|f\|_{U_\alpha} n^m N^{-m} = C_m\|f\|_{U_\alpha} N^{-\frac{\varepsilon}{1+\varepsilon}m}.$$

Consider some $\delta > 0$ and choose ε so small that $\frac{\alpha}{1+\varepsilon} > \alpha - \delta$ and m so large that $\frac{\varepsilon}{1+\varepsilon}m > \alpha - \delta$. Then we shall get

$$\begin{aligned}\|Tf - P_N\|_{L^2} &\leqslant C_m\big(N^{-(\alpha-\delta)} + n^{-\alpha}\big)\|f\|_{U_\alpha} \leqslant C_m\big(N^{-\alpha-\delta} + N^{-\frac{\alpha}{1+\varepsilon}}\big)\|f\|_{U_\alpha} \\ &\leqslant C_m N^{-(\alpha-\delta)}\|f\|_{U_\alpha}.\end{aligned}$$

(3) Obviously, $\|\mathcal{R}f\|_{L^\infty} \leqslant \|f\|_{L^\infty} \leqslant \|f\|_{U_\alpha}$. Let $\Psi = 1 - \Theta$. Note that $\mathcal{R}\mathcal{M}_n f$ is a polynomial of degree $2n$ and

$$\begin{aligned}\|\mathcal{R}f - \mathcal{R}\mathcal{M}_n f\|_{L^2}^2 &= \sum_{k\geqslant n} v_{d,k}^2 \Psi\left(\frac{k}{n}\right)^2 \|H_k^f\|_{L^2}^2 \\ &\leqslant Cn^{-2(d-2)} \sum_{k\geqslant n} \Psi\left(\frac{k}{n}\right)^2 \|H_k^f\|_{L^2}^2 \\ &= Cn^{-2(d-2)} \|f - \mathcal{M}_n f\|_{L^2}^2 \\ &\leqslant C\|f\|_{U_\alpha}^2 n^{-2(d-2+\alpha)}. \qquad \square\end{aligned}$$

Lemma 5. *Let $\beta > \alpha$. Then for every $\sigma > 0$, there exists $C = C_{\sigma,\alpha,\beta} > 0$ such that $\|f\|_{\mathcal{U}_\alpha} \leqslant C\|f\|_{L^\infty} + \sigma\|f\|_{\mathcal{U}_\beta}$.*

Proof. We have $\|f\|_{L^\infty} \leqslant C\|f\|_{L^\infty}$ as soon as $C \geqslant 1$. Now take $n \geqslant 1$. If $n^{-(\beta-\alpha)} > \sigma$, take $p_n = 0$. Then,

$$\|f - p_n\|_{L^2} \leqslant \|f\|_{L^\infty} \leqslant C\|f\|_{L^\infty} n^{-\alpha},$$

provided that $C > \sigma^{-\frac{\alpha}{\beta-\alpha}}$. If $n^{-(\beta-\alpha)} \leqslant \sigma$, choose p_n so that

$$\|f - p_n\|_{L^2} \leqslant \|f\|_{\mathcal{U}_\beta} n^{-\beta-(\beta-\alpha)} \leqslant \sigma\|f\|_{\mathcal{U}_\beta} n^{-\alpha}. \qquad \square$$

5. Iteration lemma

Lemma 6. *Fix α so large that $\mathcal{U}_\alpha \subset C^2$. Let $L > 0$ be a constant such that $\|\cdot\|_{C^2} \leqslant L\|\cdot\|_{\mathcal{U}_\alpha}$. There exist $\varepsilon_d > 0$ and $\lambda_d < 1$ with the following property. For every $\varepsilon \in (0, \varepsilon_d)$ and every function f such that $f = 1 + \varphi$, $\int \varphi = 0$, $\|\varphi\|_{L^2} \leqslant \varepsilon$, $\|\varphi\|_{\mathcal{U}_\alpha} \leqslant L^{-1}$, there exists a linear operator $T \in GL(d)$ and a positive number γ such that $\tilde{f} = \gamma\mathcal{R}(Tf)^{d-1}$ can be written as $1 + \tilde{\varphi}$ where $\int \tilde{\varphi} = 0$, $\|\tilde{\varphi}\|_{L^2} \leqslant \lambda_d\varepsilon$, $\|\tilde{\varphi}\|_{\mathcal{U}_\alpha} \leqslant L^{-1}$.*

Proof. Step 1: We show first that there exists an operator T, such that $Tf = 1 + \psi$, where $\|\psi\|_2 \leqslant \varepsilon + C\varepsilon^{\frac{d+5}{d+3}}$ and $\|H_2^\psi\|_2 \leqslant C\varepsilon^{\frac{d+5}{d+3}}$.

We shall seek T in the form $T = I + Q$ as in Lemma 2. We have

$$|Tx| = \sqrt{1 + 2(Qx, x) + \|Q\|^2} = 1 + (Qx, x) + O\big(\|Q\|^2\big).$$

Hence,

$$|Tx|^{-1} = 1 - (Qx, x) + O\big(\|Q\|^2\big).$$

Further, since $\|\varphi\|_{C^2} \leqslant L\|\varphi\|_{U_\alpha} \leqslant 1$, Lemmata 1, 2 yield

$$\big|\varphi\big(\omega_T(x)\big) - \varphi(x)\big| \leqslant C\varepsilon^{\frac{2}{d+3}}\big|\omega_T(x) - x\big| \leqslant C\varepsilon^{\frac{2}{d+3}}\|Q\|.$$

We also have

$$\begin{aligned} Tf(x) &= |Tx|^{-1}\big(1+\varphi\big(\omega_T(x)\big)\big) \\ &= \big(1-(Qx,x)+O\big(\|Q\|^2\big)\big)\big(1+\varphi(x)+O\big(\varepsilon^{\frac{2}{d+3}}\|Q\|\big)\big) \\ &= 1-(Qx,x)+\varphi(x)+O\big(\|Q\|\varepsilon^{\frac{2}{d+3}}+\|Q\|^2\big). \end{aligned} \tag{5.1}$$

Now we choose Q so that $(Qx,x)=H_2^{\varphi}(x)$. Since $\|H_2^{\varphi}\|_{L^2}\leqslant\|\varphi\|_{L^2}\leqslant\varepsilon$, and H_2^{φ} is a quadratic polynomial, we can conclude that all its coefficients do not exceed $C\varepsilon$ and thereby $\|Q\|=O(\varepsilon)$. Also, applying Lemma 1 we get $\|\varphi\|_{L^\infty}\leqslant C\varepsilon^{\frac{4}{d+3}}$. Thus, by (5.1), $Tf=1+\psi$, where $\psi=\varphi-H_2^{\varphi}+O(\varepsilon^{\frac{d+5}{d+3}})$. Note now that

$$\big\|\varphi-H_2^{\varphi}\big\|_{L^2}\leqslant\|\varphi\|_{L^2}\leqslant\varepsilon,$$

so $\|\psi\|_{L^2}\leqslant\varepsilon+O(\varepsilon^{\frac{d+5}{d+3}})$, and that $\varphi-H_2^{\varphi}$ has no spherical harmonics of degree 2 in its decomposition, so $\|H_2^{\psi}\|_{L^2}=O(\varepsilon^{\frac{d+5}{d+3}})$. Also

$$\|\psi\|_{L^\infty}\leqslant C\varepsilon^{\frac{4}{d+3}}. \tag{5.2}$$

Step 2: Now we compute $(Tf)^{d-1}$. We have

$$(Tf)^{d-1}=(1+\psi)^{d-1}=1+(d-1)\psi+\eta,$$

and (5.2) yields

$$\|\eta\|_{L^2}\leqslant C\varepsilon^{\frac{4}{d+3}}\|\psi\|_{L^2}\leqslant C\varepsilon^{\frac{d+7}{d+3}}.$$

Applying the Radon transform, we get

$$\begin{aligned} \mathcal{R}(Tf)^{d-1} &= 1+(d-1)H_0^{\psi}+H_0^{\eta}+(d-1)\mathcal{R}H_2^{\psi}+(d-1)\mathcal{R}\big(\psi-H_0^{\psi}-H_2^{\psi}\big) \\ &\quad+\mathcal{R}\big(\eta-H_0^{\eta}\big). \end{aligned}$$

Note that $(d-1)H_0^{\psi}+H_0^{\eta}$ is a constant function whose value ζ satisfies $|\zeta|\leqslant\|\psi\|_{L^2}\leqslant C\varepsilon$. We also have

$$\begin{aligned} (d-1)\big\|\mathcal{R}H_2^{\psi}\big\|_{L^2} &= \big\|H_2^{\psi}\big\|_{L^2}\leqslant C\varepsilon^{\frac{d+5}{d+3}}, \\ (d-1)\big\|\mathcal{R}\big(\psi-H_0^{\psi}-H_2^{\psi}\big)\big\|_{L^2} &\leqslant \lambda_d\|\psi\|_{L^2}, \end{aligned}$$

and

$$\big\|\mathcal{R}\big(\eta-H_0^{\eta}\big)\big\|_{L^2}\leqslant\|\eta\|_{L^2}\leqslant C\varepsilon^{\frac{d+7}{d+3}}.$$

Now take $\gamma = (1+\zeta)^{-1} = 1 + O(\varepsilon)$ and put

$$\tilde{\varphi} = \gamma\big(\mathcal{R}H_2^{\psi} + (d-1)\mathcal{R}\big(\psi - H_0^{\psi} - H_2^{\psi}\big) + \mathcal{R}\big(\eta - H_0^{\eta}\big)\big).$$

Note that

$$\|\tilde{\varphi}\|_{L^2} \leqslant \big(1+O(\varepsilon)\big)\big(\lambda_d\varepsilon + O\big(\varepsilon^{\frac{d+5}{d+3}}\big)\big) = \lambda_d\varepsilon + O\big(\varepsilon^{\frac{d+5}{d+3}}\big) < \lambda_d'\varepsilon,$$

with any $\lambda_d < \lambda_d' < 1$ provided that ε is small enough. Also $\int \tilde{\varphi} = 0$, and $\gamma\mathcal{R}(Tf)^{d-1} = 1 + \tilde{\varphi}$. At last

$$\|\tilde{\varphi}\|_{L^\infty} \leqslant C\big(\|\psi\|_{L^\infty} + \|\eta\|_{L^\infty}\big) \leqslant C\varepsilon^{\frac{4}{d+3}}.$$

Step 3: It remains to estimate $\|\tilde{\varphi}\|_{\mathcal{U}_\alpha}$. Note that $\|f\|_{\mathcal{U}_\alpha} \leqslant 2$, so applying Lemma 3, with $\delta = 1/2$, we get

$$\|Tf\|_{\mathcal{U}_{\alpha-\frac{1}{2}}} \leqslant C \quad\Rightarrow\quad \big\|(Tf)^{d-1}\big\|_{\mathcal{U}_{\alpha-\frac{1}{2}}} \leqslant C' \quad\Rightarrow\quad \big\|\mathcal{R}(Tf)^{d-1}\big\|_{\mathcal{U}_\beta} \leqslant C''$$
$$\Rightarrow\quad \|\tilde{\varphi}\|_{\mathcal{U}_\beta} \leqslant C''',$$

where $\beta = \alpha - \frac{1}{2} + d - 2 > \alpha$. Now choose $\sigma > 0$ so that $C'''\sigma \leqslant \frac{1}{2L}$. Then, by Lemma 5,

$$\|\tilde{\varphi}\|_{\mathcal{U}_\alpha} \leqslant \sigma C''' + C_{\sigma,\alpha,\beta}C'\varepsilon^{\frac{4}{d+3}} \leqslant \frac{1}{L},$$

provided that ε is small enough. □

6. Smoothing

Fix $\beta > \alpha > 0$. Let $f = 1 + \varphi$, $\|\varphi\|_{L^\infty} < \varepsilon < 1/2$. Define the sequence f_k recursively by $f_0 = f$, $f_{k+1} = \mathcal{R}f_k^{d-1}$. Using Lemma 3, we can conclude that $f_k \in \mathcal{U}_\beta$ for sufficiently large k and $\|f_k\|_{\mathcal{U}_\beta} \leqslant C(k)$. Also, it is easy to show by induction that

$$(1-\varepsilon)^{(d-1)^k} \leqslant f_k \leqslant (1+\varepsilon)^{(d-1)^k}.$$

Let $\mu = \int f_k$. If $\varepsilon > 0$ is sufficiently small, then $|\mu - 1|$ is small and $\mu^{-1}f_k = 1 + \psi$ where $\int \psi = 0$ and $\|\psi\|_{L^\infty}$ is small. Note that

$$\|\psi\|_{\mathcal{U}_\beta} \leqslant 1 + \mu^{-1}\|f_k\|_{\mathcal{U}_\beta} \leqslant C'(k),$$

and, thereby, by Lemma 5, $\|\psi\|_{\mathcal{U}_\alpha}$ is also small ($\|\psi\|_{\mathcal{U}_\beta}$ is bounded by a fixed constant and $\|\psi\|_{L^\infty} \to 0$ as $\varepsilon \to 0$). Applying this observation to the function ρ_K, we conclude that if K is sufficiently close to B_d, then, after proper normalization, $\rho_{\mathcal{I}^k K}$ can be written as $1 + \varphi$ with $\|\varphi\|_{\mathcal{U}_\alpha}$ as small as we want.

7. The end of the proof

Now we choose ε so small that the smoothing part results in a body K for which ρ_K satisfies the assumptions of Lemma 6. Then ρ_{K_1}, where $K_1 = \gamma\mathcal{I}TK$ satisfies the assumptions of Lemma 6 with $\lambda\varepsilon$ instead of ε. Note that $K_1 \stackrel{\mathbb{S}_d}{=} \mathcal{I}K$. Applying Lemma 6 again, we get a body $K_2 \stackrel{\mathbb{S}_d}{=} \mathcal{I}^2K$ such that ρ_{K_2} satisfies the assumption of Lemma 6 with $\lambda^2\varepsilon$ instead of ε and so on.

In particular, it means that

$$\|\rho_{K_m} - 1\|_{L^2} \leqslant \lambda^m\varepsilon \to 0 \quad \text{as } m \to \infty$$

and $\|\rho_{K_m}\|_{C^2} \leqslant 2$.

This is enough to conclude that

$$d_{BM}(K_m, B_d) = d_{BM}\big(\mathcal{I}^m K, B_d\big) \to 0 \quad \text{as } m \to \infty.$$

Appendix A

Proposition. *Consider $\Theta \in C_0^\infty(\mathbb{R})$. Then the operator $\mathcal{M}_n^\Theta$ defined in* (4.1) *is bounded in L^p, for all $1 \leqslant p \leqslant \infty$, i.e.*

$$\big\|\mathcal{M}_n^\Theta f\big\|_{L^p(S^{d-1})} \leqslant C\|f\|_{L^p(S^{d-1})}. \tag{A.1}$$

The proposition is well known to the specialists but to make the paper self-contained, we present its proof below.

We start the proof with some auxiliary lemmata. We assume below that the measure σ on the sphere is normalized so that the total measure of the sphere is one.

For every $z \in \mathbb{C}$ such that $|z| < 1$, define the function $P_z(\mathbf{x}, \mathbf{y}) : S^{d-1} \times S^{d-1} \to \mathbb{C}$ by

$$P_z(\mathbf{x}, \mathbf{y}) := \frac{1 - z^2}{(1 + z^2 - 2z(\mathbf{x} \cdot \mathbf{y}))^{d/2}}, \quad z \in \mathbb{C},\ |z| < 1, \tag{A.2}$$

where for odd d we pick the branch of an analytic function

$$z \to g(z) = \big(1 + z^2 - 2z(\mathbf{x} \cdot \mathbf{y})\big)^{d/2}$$

in such a way that $g(\mathbb{R}_+) \subset \mathbb{R}_+$.

Lemma 7. *For all $x, y \in S^{d-1}$, and $|z| < 1$*

$$\big|P_z(\mathbf{x}, \mathbf{y})\big| \leqslant 2 \cdot 3^d \left(\frac{|1 - z|}{1 - |z|}\right)^{d+1} P_{|z|}(\mathbf{x}, \mathbf{y}).$$

Proof. For $\beta \in \mathbb{C}$, $|\beta| = 1$, we have

$$\frac{||z| - \beta|}{|z - \beta|} \leqslant 1 + \frac{|z - |z||}{|z - \beta|} \leqslant 1 + \frac{|z - |z||}{||z| - 1|} \leqslant \frac{|z - |z|| + ||z| - 1|}{1 - |z|} \leqslant 3\frac{|1 - z|}{1 - |z|}.$$

We also have

$$\frac{|1-z^2|}{1-|z|^2} \leqslant 2\frac{|1-z|}{1-|z|}.$$

Since

$$1+z^2-2z(\mathbf{x}\cdot\mathbf{y}) = (z-\alpha)(z-\bar{\alpha}), \quad \text{for } \alpha = \mathbf{x}\cdot\mathbf{y} + i\sqrt{1-(\mathbf{x}\cdot\mathbf{y})^2},$$

we conclude

$$\frac{|P_z(\mathbf{x},\mathbf{y})|}{P_{|z|}(\mathbf{x},\mathbf{y})} = \frac{|1-z^2||(|z|-\alpha)(|z|-\bar{\alpha})|^{d/2}}{|1-|z|^2||(z-\alpha)(z-\bar{\alpha})|^{d/2}} \leqslant 2\cdot 3^d \left(\frac{|1-z|}{1-|z|}\right)^{d+1}. \qquad \square$$

Lemma 8. *Let* $z\in\mathbb{C}$, $0<\operatorname{Im} z<2$, *and let* $n\in\mathbb{N}$. *Then,*

$$\left\| P_{e^{iz/n}}(\mathbf{x},\cdot)\right\|_{L^1(S^{d-1})} \leqslant 2^{d+2}\cdot 3^d \left(\frac{|z|}{\operatorname{Im} z}\right)^{d+1}.$$

Proof. Put $\xi = iz/n$. Then,

$$\begin{aligned}\frac{|1-e^{\xi}|}{1-e^{\xi}} &\leqslant 1+\frac{|e^{\xi}-e^{\operatorname{Re}\xi}|}{1-e^{\operatorname{Re}\xi}} \leqslant 1+\frac{e^{\operatorname{Re}\xi}|\operatorname{Im}\xi|}{1-e^{\operatorname{Re}\xi}} = 1+\frac{|\operatorname{Im}\xi|}{e^{-\operatorname{Re}\xi}-1} \leqslant 1+\frac{|\operatorname{Im}\xi|}{|\operatorname{Re}\xi|}\\ &\leqslant \frac{2|\xi|}{|\operatorname{Re}\xi|} = 2\frac{|z|}{\operatorname{Im} z}.\end{aligned}$$

Now by Lemma 7,

$$\left|P_{e^{iz/n}}(\mathbf{x},\mathbf{y})\right| \leqslant 2\cdot 3^d\left(\frac{|1-e^{iz/n}|}{1-|e^{iz/n}|}\right)^{d+1} P_{|e^{iz/n}|}(\mathbf{x},\mathbf{y}) \leqslant 2^{d+2}\cdot 3^d\left(\frac{|z|}{\operatorname{Im} z}\right)^{d+1} P_{|e^{iz/n}|}(\mathbf{x},\mathbf{y}).$$

It remains to use $\|P_{|e^{iz/n}|}(\mathbf{x},\cdot)\|_{L^1(S^{d-1})} = 1$. $\square$

Let $S(\mathbb{R})$ be the Schwartz space. To prove (A.1), write

$$\Theta\left(\frac{k}{n}\right) = \int\limits_{\mathbb{R}} \psi(x)e^{ikx/n}\,dx, \tag{A.3}$$

where $\psi\in S(\mathbb{R})$ is the Fourier transform of some C_0^∞ extension of Θ to the entire real line.

Using the Stokes formula, we can rewrite the last integral as

$$2i\int\limits_{\operatorname{Im} z>0} \bar{\partial}\Psi(z)e^{ikz/n}\,dA(z),$$

where Ψ is any reasonable extension of ψ to the upper half-plane. To make this representation useful, we shall need the following lemma:

Lemma 9. *For any $\psi \in S(\mathbb{R})$ there exists an extension $\Psi(z)$, $\operatorname{Im} z \geqslant 0$, $\Psi|_{\mathbb{R}}(x) = \psi(x)$, such that*

$$\int\limits_{\operatorname{Im} z>0} \left|\bar\partial \Phi(z)\right| \left(\frac{|z|}{\operatorname{Im} z}\right)^{d+1} dA(z) < \infty.$$

Let us first show that Lemma 9 gives $\|M_n^\Theta\|_{L_p \to L_p} < \infty$. Indeed, using (A.3), we can calculate the kernel K_n of the operator M_n^Θ,

$$\mathcal{M}_n^\Theta f = \sum_{k=0}^{\infty} \Theta(k/n) H_k^f = \int\limits_{\mathbb{R}} \psi(x) \sum_{k=0}^{\infty} e^{ikx/n} H_k^f \, dx$$
$$= 2i \int\limits_{\operatorname{Im} z>0} \bar\partial \Psi(z) \sum_{k=0}^{\infty} e^{ikz/n} H_k^f \, dA(z).$$

Now note that

$$\sum_{k=0}^{\infty} e^{ikz/n} H_k^f(\mathbf{x}) = \int\limits_{S^{d-1}} P_{e^{iz/n}}(\mathbf{x}, \mathbf{y}) f(\mathbf{y}) \, d\sigma(\mathbf{y}).$$

So,

$$K_n = 2i \int\limits_{\operatorname{Im} z>0} \bar\partial \Psi(z) P_{e^{iz/n}} \, dA(z).$$

Since $\|K_n(\mathbf{x}, \cdot)\|_{L^1(S^{d-1})} \leqslant C$, we have $\|K_n(\cdot, \mathbf{y})\|_{L^1(S^{d-1})} \leqslant C$ by symmetry. Now (A.1) follows from the Schur test.

Let us now prove Lemma 9:

Proof. We define

$$\Psi(x+iy) = \eta(y)\Psi_0(x+iy), \qquad \Psi_0(x+iy) = \sum_{k=0}^{d+1} \psi^{(k)}(x)(iy)^k/k!,$$

where $\eta : [0, \infty) \to [0, 1]$ is infinitely differentiable, $\eta(y) = 1$ for $0 \leqslant y \leqslant 1$, and $\eta(y) = 0$ for $y \geqslant 2$. Observe that

$$\left|2\bar\partial \Psi(x+iy)\right| = \left|(\partial/\partial x + i\partial/\partial y)\Psi(x+iy)\right|$$
$$\leqslant \left|\Psi_0(x+iy)\right|\left|\eta'(y)\right| + \left|\eta(y)\right|\left|\psi^{(d+2)}(x)(iy)^{d+1}/(d+1)!\right|.$$

Hence,

$$\int\limits_{\operatorname{Im} z>0} \left|\bar\partial \Phi(z)\right| \left(\frac{|z|}{\operatorname{Im} z}\right)^{d+1} dA(z)$$
$$\leqslant 2 \int\limits_{\operatorname{Im} z\leqslant 2} \left|\Psi_0(x+iy)\right| |z|^{d+1}\, dA(z) + \frac{1}{(d+1)!} \int\limits_{\operatorname{Im} z\leqslant 2} \left|\psi^{(d+2)}(x)\right| |z|^{d+1}\, dA(z) \leqslant C,$$

and we are done, since $\psi \in S(\mathbb{R})$. □

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