THE k-DIMENSIONAL RADON TRANSFORM ON THE n-SPHERE AND RELATED WAVELET TRANSFORMS

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Abstract. Continuous wavelet transforms, associated with the k-dimensional spherical Radon transform Rf on the n-dimensional unit sphere $S^n, n \geq 2$, are introduced. It is assumed that $f \in L^p(S^n), \quad 1 \leq p < \infty$, or $f \in C(S^n)$. For the operator R and for its left inverse R^{-1} explicit representations are given in terms of the relevant continuous wavelet transforms.

1. Introduction and main results

Let $S^n \subset \mathbb{R}^{n+1}$ be the unit sphere and let Ξ be the set of all k-dimensional totally geodesic submanifolds of S^n, $1 \leq k \leq n-1$. Given a continuous function f on S^n, consider the k-dimensional spherical Radon transform

$$Rf(\xi) = \int\limits_{\xi} f(x) dm(x), \qquad \xi \in \Xi, \tag{1.1}$$

where dm is the natural measure on ξ induced by the Lebesgue measure on S^n and normalized so that $\int\limits_{\xi} dm(x) = 1$. This transform was studied by different authors (see, e.g., [1], [6], [7], [8], [11], [13], [24]) and plays an important role in geometrical problems for convex bodies ([2], [9], [10], [26]). In the present article we develop a wavelet approach to the inversion problem for (1.1). The following approaches to this problem, which differ from ours, are known in the literature.

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S. Helgason [11] suggested two inversion procedures based on the duality principle. The first formula reads ([11], p. 93)

$$f = P(\Delta) R^* R f \ , \tag{1.2}$$

and works only for k even. Here $P(\Delta)$ is a certain polynomial of the Laplace-Beltrami operator on S^n, R^* is the dual transform which designates the average over the set of all ξ passing through x. The second inversion formula ([11], p. 99), which works for all $1 \le k \le n-1$, is as follows:

$$f(x) = \frac{c}{2} \Big[\Big(\frac{d}{d(u^2)} \Big)^k \int_0^u (R^*_{\cos^{-1}(v)} R f)(x) v^k (u^2 - v^2)^{k/2-1} dv \Big] \Big|_{u=1}. \tag{1.3}$$

Here $c^{-1} = (k-1)! \, \sigma_{k+1}/2^{k+1}$, $R^*_{\cos^{-1}(v)} R f$ is the average of Rf over the set of all ξ at the distance $p = \cos^{-1}(v)$ from x. This formula is based on the observation that $R^*_p R f$ can be written as a fractional integral of order $k/2$ of a certain average of f ([11], pp. 98, 99).

R. Strichartz ([24], p. 725) proved the following inversion formula:

$$f = c_k^{-1} \mathcal{R} (-\widetilde{\Delta})^{k/2} E_{-1} R^* R f. \tag{1.4}$$

Here $c_k = 2^k \Gamma(k+1/2) \Gamma(n/2) / \sqrt{\pi} \, \Gamma(n-k/2)$, $\mathcal{R}$ denotes the restriction operator from functions on $\mathbb{R}^{n+1}$ to S^n; $E_{-1} f$ is the extension of f to a homogeneous function of degree -1; $\widetilde{\Delta}$ denotes the Laplacian on $\mathbb{R}^{n+1}$.

T. Kakehi ([13], p. 319) showed how to use (1.2) in the case of k odd. He constructed an operator L such that LRf is the $(k+1)$-dimensional spherical Radon transform and then applied (1.2) to LRf.

One should mention the papers by I. M. Gelfand, S. G. Gindikin, M. I.Graev [6], S. G. Gindikin [7], E. L. Grinberg [8] (and references therein), where inversion formulas are given in the context of projective spaces. A series of inversion formulas like (1.2), which work for all $1 \le k \le n-1$ and involve fractional integrals associated with Rf, was obtained by B. Rubin [19]. There is a number of remarkable papers by F. B. Gonzalez, E. L. Grinberg, T. Kakehi, E. T. Quinto and others devoted to the range characterization of Rf (see [11] for references).

All aforementioned methods work well for smooth functions. Applicability of these methods to nonsmooth $f \in L^p(S^n)$ or $f \in C(S^n)$ is a matter of special investigation (see discussion in Section 5). In the case $k = n-1$ the Radon transform (1.1) for nonsmooth f was inverted in [18] in terms of the relevant continuous wavelet transforms.

In the present paper we extend the results from [18] to all $1 \le k \le n-1$. Our inversion formula has a simple form and agrees with the general philosophy developed in [17]. Following [6] and [24], we regard (1.1) as a function $Rf(v)$ on the Stiefel manifold $V = V_{n+1,n-k}$ of all orthonormal $(n-k)$-frames in $\mathbb{R}^{n+1}$. Here $v = (v^1, \dots, v^{n-k}) \in V$ is an $(n+1)\times(n-k)$ matrix with pairwise orthogonal unit column vectors $v^1, \dots, v^{n-k} \in \mathbb{R}^{n+1}$.

For $f \in C(S^n)$ the Radon transform (1.1) can be written as

$$Rf(v) = \frac{1}{|S^k|}\int\limits_{S^k} f(r_v\eta)d\eta, \quad v \in V. \tag{1.5}$$

Here and on $d\eta$ is the natural Lebesgue measure on S^k, $r_v \in SO(n+1)$, $r_v v_0 = v$, $v_0 = (e_{k+2}, \dots, e_{n+1})$ is the coordinate $(n-k)$-frame, $e_i = (0, \dots, \overset{i}{1}, 0, \dots, 0) \in \mathbb{R}^{n+1}$.

Let us state our main results. For $f \in L^1(S^n)$ and $\varphi \in L^1(V)$, the intertwining continuous wavelet transforms, associated with (1.1) are defined by

$$Wf(v,t) = t^{k-n}\int\limits_{S^n} f(x)w(|x\cdot v|/t)dx, \qquad v \in V,\ t>0, \tag{1.6}$$

$$\overset{*}{W}\varphi(x,t) = t^{k-n}\int\limits_{V} \varphi(v)w(|x\cdot v|/t)dv, \qquad x \in S^n,\ t>0. \tag{1.7}$$

Here w is a sufficiently nice "wavelet function" on $\mathbb{R}_+ = [0,\infty)$, and $x\cdot v$ is an $(n-k)$-vector defined by

$$x\cdot v = [x_1, \dots, x_{n+1}]\begin{bmatrix} v_1^1 \dots v_1^{n-k} \\ \dots\dots\dots \\ v_{n+1}^1 \dots v_{n+1}^{n-k}\end{bmatrix} = [\sum_{j=1}^{n+1} x_j v_j^1, \dots, \sum_{j=1}^{n+1} x_j v_j^{n-k}];$$

dx and dv are the corresponding $SO(n+1)$-invariant measures on S^n and V, normalized so that $\sigma_n = |S^n| = 2\pi^{(n+1)/2}/\Gamma((n+1)/2)$; $|V| = \sigma_n\sigma_{n-1}\cdots\sigma_{k+1}$ (see e.g., [12], [21, p. 208]). For $k=n-1$ transforms (1.6) and (1.7) are identical and coincide with those in [18].

Theorem A. *Let*

$$\int_0^\infty \tau^{j+n-k-1} w(\tau) d\tau = 0 \quad \forall j = 0, 2, 4, \dots, 2[k/2], \tag{1.8}$$

($[k/2]$ *is the integral part of* $k/2$),

$$\int_1^\infty \tau^{\beta+n-k-1} |w(\tau)| d\tau < \infty \qquad for\ some\ \ \beta > k. \tag{1.9}$$

Suppose that $\varphi(v) = Rf(v)$, $v \in V$, *where* f *is an even function belonging to* $L^p(S^n)$, $1 \le p < \infty$, *or* $f \in C(S^n)$. *Then*

$$\int_0^\infty \frac{(\overset{*}{W}\varphi)(x,t)}{t^{k+1}} dt = \lim_{\varepsilon \to 0} \int_\varepsilon^\infty \frac{(\overset{*}{W}\varphi)(x,t)}{t^{k+1}} dt = \alpha\, f(x), \tag{1.10}$$

$$\alpha = \frac{|V|\, \Gamma(\frac{n+1}{2})}{\pi^{1/2}\Gamma(\frac{n-k}{2})} \begin{cases} \Gamma\left(-k/2\right) \int_0^\infty \tau^{n-1} w(\tau) d\tau & if\ \ k\ \ is\ odd, \\ \frac{2(-1)^{1+k/2}}{(k/2)!} \int_0^\infty \tau^{n-1} w(\tau) \log \tau d\tau & if\ \ k\ \ is\ even. \end{cases} \tag{1.11}$$

The limit in (1.10) is understood in the L^p*-norm and in the a.e. sense. For* $f \in C(S^n)$, *the limit in (1.10) is interpreted in the sup-norm.*

The next statement contains an analogue of the Calderón reproducing formula for the k-dimensional spherical Radon transform.

Theorem B. *Let*

$$\int_0^\infty w(\tau) \tau^{n-k-1} d\tau = 0, \qquad \int_0^\infty |w(\tau) \log \tau| \tau^{n-k-1} d\tau < \infty. \tag{1.12}$$

If $f \in L^p(S^n)$, $1 \le p < \infty$, *or* $f \in C(S^n)$, *then*

$$\int_0^\infty \frac{(Wf)(v,t)}{t} dt = \lim_{\varepsilon \to 0} \int_\varepsilon^\infty \frac{(Wf)(v,t)}{t} dt = \beta\, Rf(v),$$
$$\beta = \frac{4\pi^{(n-k)/2}}{\Gamma\left((n-k)/2\right)} \int_0^\infty w(\tau) \tau^{n-k-1} \log \frac{1}{\tau} d\tau, \tag{1.13}$$

where the limit is understood in the L^p-*norm or in the sup-norm.*

Remark. One can define the Radon transform (1.1) as a function on the dual Stiefel manifold $\tilde{V} = V_{n+1,k+1}$:

$$\tilde{R}f(u) = \frac{1}{|S^k|} \int f(\tilde{r}_u x) d\sigma(x), \quad u \in \tilde{V}.$$

Here $\tilde{r}_u \in SO(n+1) : \tilde{r}_u u_0 = u, u_0 = (e_1, \ldots, e_{k+1}) \in \tilde{V}$ is the coordinate $(k+1)$-frame. It is clear that $\tilde{R}f(u) = Rf(v)$ provided that columns of the $(n+1)\times(n+1)$-matrix (u, v) generate an orthonormal basis of $\mathbb{R}^{n+1}$. One can reformulate Theorems A and B in terms of $\tilde{R}f$. In this case the intertwining continuous wavelet transforms, associated with $\tilde{R}f$, are defined by:

$$Uf(u,t) = t^{k-n} \int\limits_{S^n} f(x) w(\sqrt{1-|x\cdot u|^2}/t) dx, \qquad u \in \tilde{V}, \;\; t > 0, \tag{1.14}$$

$$\overset{*}{U}f(x,t) = t^{k-n} \int\limits_{\tilde{V}} \varphi(u) w(\sqrt{1-|x\cdot u|^2}/t) du, \qquad x \in S^n, \;\; t > 0. \tag{1.15}$$

2. Preliminaries

Lemma 2.1. *Let $f \in L^1(S^n),\ f \geq 0$. Then*

$$\|Rf\|_{L^1(V)} = \sigma_n^{-1} |V| \; \|f\|_{L^1(S^n)}. \tag{2.1}$$

Proof. By (1.5),

$$\|Rf\|_{L^1(V)} = \frac{1}{\sigma_k} \int\limits_{S^k} d\eta \int\limits_{V} f(r_v \eta) dv = \frac{1}{\sigma_k} \int\limits_{S^k} d\eta \int\limits_{V} dv \int\limits_{SO(n+1)} f(r_{\gamma v}\eta) d\gamma$$

$$= \frac{|V|}{\sigma_k} \int\limits_{S^k} d\eta \int\limits_{SO(n+1)} f(\alpha\eta) d\alpha = \frac{|V|}{\sigma_n} \|f\|_{L^1(S^n)}. \qquad \square$$

We shall use the bispherical coordinates on S^n (see, e.g., [25], p. 12) defined by

$$x = \xi \cos\theta + \eta \sin\theta, \; \xi \in S^{n-k-1} \subset \mathbb{R}^{n-k}, \; \eta \in S^k \subset \mathbb{R}^{k+1}, \; 0 \leq \theta \leq \frac{\pi}{2}, \tag{2.2}$$

$$\mathbb{R}^{k+1} = span\ (e_1, \ldots, e_{k+1}); \qquad \mathbb{R}^{n-k} = span\ (e_{k+2}, \ldots, e_{n+1}).$$

According to (2.2),

$$dx = \sin^k\theta \, \cos^{n-k-1}\theta \; d\theta d\xi d\eta.$$

For $v \in V, \; \theta \in [0, \pi/2]$, we introduce a mean-value operator

$$M_{\cos\theta} f(v) = \frac{1}{\sigma_{n-k-1}\sigma_k} \int\limits_{S^{n-k-1}} d\xi \int\limits_{S^k} f(r_v(\xi\cos\theta + \eta\sin\theta)) d\eta, \tag{2.3}$$

so that

$$\int\limits_{S^n} f(x) dx = \sigma_{n-k-1}\sigma_k \int\limits_0^{\pi/2} \sin^k\theta \, \cos^{n-k-1}\theta \; M_{\cos\theta} f(v) d\theta.$$

In particular, for $\theta = \pi/2$, we have $M_0 f(v) = Rf(v)$

Lemma 2.2. *Let $f \in L^p(S^n)$, $1 \le p < \infty$. Then*

$$\sup_{\theta \in [0,\pi/2]} \|M_{\cos\theta} f\|_{L^p(V)} \le (|V|/\sigma_n)^{1/p} \, \|f\|_{L^p(S^n)}, \tag{2.4}$$

$$\lim_{\theta \to \pi/2} \|M_{\cos\theta} f - Rf\|_{L^p(V)} = 0. \tag{2.5}$$

If $f \in C(S^n)$, then $M_{\cos\theta} f \to Rf$ as $\theta \to \pi/2$ uniformly on S^n.

Proof. By (2.3) and Minkowski's inequality,

$$\|M_{\cos\theta} f\|_{L^p(V)} \le \frac{|V|^{1/p}}{\sigma_{n-k-1}\sigma_k} \int\limits_{S^{n-k-1}} d\xi \int\limits_{S^k} d\eta \Big(\int\limits_{SO(n+1)} |f(r_{gv_0}(\xi\cos\theta + \eta\sin\theta))|^p dg \Big)^{1/p}$$
$$= (|V|/\sigma_n)^{1/p} \, \|f\|_{L^p(S^n)}.$$

Furthermore, $\|M_{\cos\theta} f - Rf\|_{L^p(V)}$ does not exceed

$$\frac{|V|^{1/p}}{\sigma_{n-k-1}\sigma_k} \int\limits_{S^{n-k-1}} d\xi \int\limits_{S^k} d\eta \Big(\int\limits_{SO(n+1)} |f(r_{gv_0}(\xi\cos\theta + \eta\sin\theta)) - f(r_{gv_0}\eta)|^p dg \Big)^{1/p}.$$

The last expression tends to 0 as $\theta \to \pi/2$ provided $f \in C(S^n)$. Since $C(S^n)$ is dense in $L^p(S^n)$, $1 \le p < \infty$, we have (2.5). The proof of $M_{\cos\theta} f \to Rf$ as $\theta \to \pi/2$ uniformly on S^n is similar with $\|\cdot\|_p$ replaced by the corresponding sup-norm. □

Consider the intertwining operators which were introduced in [19]:

$$Af(v) = \int\limits_{S^n} a(|x \cdot v|) f(x) dx, \qquad \overset{*}{A}\varphi(x) = \int\limits_{V} a(|x \cdot v|)\varphi(v) dv.$$

Lemma 2.3. *Let $f \in L^p(S^n)$, $\varphi \in L^p(V)$, $1 \le p \le \infty$. Then*

$$\|Af\|_{L^p(V)} \le \sigma_{n-k-1}\sigma_k \, (|V|/\sigma_n)^{1/p} \, \|a\| \, \|f\|_{L^p(S^n)}, \tag{2.6}$$

$$\|\overset{*}{A}\varphi\|_{L^p(S^n)} \le \sigma_{n-k-1}\sigma_k \, (|V|/\sigma_n)^{1-1/p} \, \|a\| \, \|\varphi\|_{L^p(V)}, \tag{2.7}$$

where $\|a\| = \int\limits_0^{\pi/2} \sin^k\theta \, \cos^{n-k-1}\theta \, |a(\cos\theta)| d\theta.$

Proof. Replacing $x \to r_v x$ and passing to the bispherical coordinates (2.2), we have:

$$Af(v) = \sigma_{n-k-1}\sigma_k \int\limits_0^{\pi/2} a(\cos\theta) \, \sin^k\theta \, \cos^{n-k-1}\theta M_{\cos\theta} f(v) d\theta. \tag{2.8}$$

Now (2.6) follows from (2.8), (2.4) and Minkowski's inequality; (2.7) follows from (2.6) by duality: $\int_{S^n} f(x) \overset{*}{A}\varphi(x) dx = \int_V \varphi(v) Af(v) dv$. □

Given $x \in S^n$ and $t \in (-1,1)$, denote

$$\mathbb{M}_t f(x) = \frac{(1-t^2)^{(1-n)/2}}{\sigma_{n-1}} \int\limits_{\{y \in S^n :\, x \cdot y = t\}} f(y) d\sigma(y), \tag{2.9}$$

where $x \cdot y = x_1 y_1 + \cdots + x_{n+1} y_{n+1}$ is a usual inner product in $\mathbb{R}^{n+1}$, $d\sigma(y)$ designates the corresponding Lebesgue measure induced by that on S^n. The integral (2.9) is the mean value of f on the planar section of S^n by the hyperplane $x \cdot y = t$.

Lemma 2.4. (cf. [11], p. 97). *Let $r_x \in SO(n+1)$ be such that $r_x e_{n+1} = x \in S^n$. Suppose that $f \in L^1(S^n)$. Then*

$$\int\limits_{SO(n)} Rf(r_x \rho r_x^{-1} \gamma v_0) d\rho = \frac{1}{\sigma_k} \int\limits_{S^k} \mathbb{M}_{x \cdot \gamma\eta} f(x) d\eta \tag{2.10}$$

for a. e. $x \in S^n$ and for any $\gamma \in SO(n+1)$.

Proof. We start with the obvious formula

$$\int\limits_{SO(n)} \varphi(\rho z) d\rho = \mathbb{M}_{z_{n+1}} \varphi(e_{n+1}), \qquad z \in S^n,$$

and set $z = r_x^{-1} \gamma \eta$, $\varphi(y) = f(r_x y)$. This yields

$$\int\limits_{SO(n)} f(r_x \rho r_x^{-1} \gamma \eta) d\rho = \mathbb{M}_{x \cdot \gamma\eta} f(x). \tag{2.11}$$

Since $Rf(\alpha v) = 1/\sigma_k \int_{S^k} f(\alpha r_v \eta) d\eta \ \forall \alpha \in SO(n+1)$, by (1.5) and (2.11) we obtain

$$\int\limits_{SO(n)} Rf(r_x \rho r_x^{-1} \gamma v_0) d\rho = \frac{1}{\sigma_k} \int\limits_{S^k} d\eta \int\limits_{SO(n)} f(r_x \rho r_x^{-1} \gamma \eta) d\rho = \frac{1}{\sigma_k} \int\limits_{S^k} \mathbb{M}_{x \cdot \gamma\eta} f(x) d\eta.$$

□

3. Proof of Theorem A.

In the following we deal with the Riemann-Liouville fractional integrals

$$(I^\alpha_{0+} \psi)(t) = \frac{1}{\Gamma(\alpha)} \int\limits_0^t \frac{\psi(\tau) d\tau}{(t-\tau)^{1-\alpha}}, \qquad (I^\alpha_{1-} \psi)(t) = \frac{1}{\Gamma(\alpha)} \int\limits_t^1 \frac{\psi(\tau) d\tau}{(\tau - t)^{1-\alpha}}, \quad Re\alpha > 0.$$

The next statement generalizes Lemma 2.3 of [18].

Lemma 3.1. *Let a be such that $\int\limits_0^1 z^{n-k-1}(1-z^2)^{(k-1)/2}|a(z)|dz<\infty$, and let $f\in L^1(S^n)$ be even. Then for a.e. $x\in S^n$,*

$$\overset{*}{A}Rf(x)\equiv\int\limits_V a(|x\cdot v|)Rf(v)dv=c_{k,n}\int\limits_0^1 z^{n-k-1}a(z)(I_{1-}^{k/2}g)(z^2)dz, \tag{3.1}$$

where $g(u)=(1-u)^{-1/2}\mathbb{M}_{\sqrt{1-u}}f(x)$,

$$c_{k,n}=\frac{2|V|\,\Gamma((n+1)/2)}{\sqrt{\pi}\,\Gamma((n-k)/2)}. \tag{3.2}$$

Proof. By (2.1) and (2.7) the expression $\overset{*}{A}Rf(x)$ is well-defined for a.e. $x\in S^n$. Fix any such x, and replace $v\to r_xv$, where $r_x\in SO(n+1), r_xe_{n+1}=x$. We get

$$\overset{*}{A}Rf(x)=\int\limits_V a(|e_{n+1}\cdot v|)Rf(r_xv)dv=|V|\int\limits_{SO(n+1)}a(|e_{n+1}\cdot\gamma v_0|)Rf(r_x\gamma v_0)d\gamma.$$

Next we replace $\gamma\to\rho r_x^{-1}\gamma, \rho\in SO(n)$, and integrate in ρ. Since $e_{n+1}\cdot\rho r_x^{-1}\gamma v_0=e_{n+1}\cdot r_x^{-1}\gamma v_0$, we use (2.10) to obtain

$$\begin{aligned}\overset{*}{A}Rf(x)&=|V|\int\limits_{SO(n+1)}a(|e_{n+1}\cdot r_x^{-1}\gamma v_0|)d\gamma\int\limits_{SO(n)}Rf(r_x\rho r_x^{-1}\gamma v_0)d\rho\\ &=\frac{|V|}{\sigma_k}\int\limits_{SO(n+1)}a(|e_{n+1}\cdot r_x^{-1}\gamma v_0|)d\gamma\int\limits_{S^k}\mathbb{M}_{x\cdot\gamma\eta}f(x)d\eta \qquad (r_x^{-1}\gamma=\alpha)\\ &=\frac{|V|}{\sigma_k}\int\limits_{SO(n+1)}a(|e_{n+1}\cdot\alpha v_0|)d\alpha\int\limits_{S^k}\mathbb{M}_{e_{n+1}\cdot\alpha\eta}f(x)d\eta\\ &=\frac{|V|}{\sigma_k\sigma_n}\int\limits_{S^n}a(|y\cdot v_0|)dy\int\limits_{S^k}\mathbb{M}_{y\cdot\eta}f(x)d\eta=\frac{2|V|}{\sigma_k\sigma_n}\int\limits_{S^n}a(|y\cdot v_0|)dy\int\limits_{y\cdot\eta>0}\mathbb{M}_{y\cdot\eta}f(x)d\eta.\end{aligned}$$

Passing to bispherical coordinates $y=\xi'\cos\theta+\eta'\sin\theta$, we have $|y\cdot v_0|=\cos\theta$, $y\cdot\eta=(\eta'\cdot\eta)\sin\theta$. This yields

$$\int\limits_{y\eta>0}\mathbb{M}_{y\cdot\eta}f(x)d\eta=\int\limits_{\eta\cdot\eta'>0}\mathbb{M}_{(\eta\cdot\eta')\sin\theta}f(x)d\eta=\sigma_{k-1}\int\limits_0^1(1-\tau^2)^{k/2-1}\mathbb{M}_{\tau\sin\theta}f(x)d\tau,$$

and therefore $\overset{*}{A}Rf(x)$ is equal to

$$\begin{aligned}&\frac{2|V|\sigma_{k-1}\sigma_{n-k-1}}{\sigma_n}\int\limits_0^{\pi/2}\sin^k\theta\,\cos^{n-k-1}\theta\;a(\cos\theta)d\theta\int\limits_0^1(1-\tau^2)^{k/2-1}\mathbb{M}_{\tau\sin\theta}f(x)d\tau\\ &=\frac{2|V|\sigma_{k-1}\sigma_{n-k-1}}{\sigma_n}\int\limits_0^1 z^{n-k-1}(1-z^2)^{(k-1)/2}a(z)dz\int\limits_0^1(1-\tau^2)^{k/2-1}\mathbb{M}_{\tau\sqrt{1-z^2}}f(x)d\tau.\end{aligned}$$

Finally we replace $\tau\sqrt{1-z^2}$ by $\sqrt{1-u}$ and obtain (3.1). $\square$

Consider the dual wavelet transform (1.7) applied to $\varphi = Rf$. By setting

$$w_1(\tau) = \tau^{(n-k)/2-1} w(\sqrt{\tau}), \qquad h = I_{0+}^{k/2} w_1,$$

owing to Lemma 3.1, we get

Corollary 3.2.

$$(\overset{*}{W} Rf)(x,t) = \frac{c_{k,n}\, t^{k-2}}{2} \int\limits_0^1 g(u) h\left(\frac{u}{t^2}\right) du, \tag{3.3}$$

g and $c_{k,n}$ being the same as in Lemma 3.1.

Proof. By (1.7) and (3.1),

$$\begin{aligned}
(\overset{*}{W} Rf)(x,t) &= \frac{c_{k,n}}{t^{n-k}} \int\limits_0^1 z^{n-k-1} w\left(\frac{z}{t}\right) (I_{1-}^{k/2} g)(z^2) dz \\
&= \frac{c_{k,n}}{\Gamma(k/2)} \int\limits_0^{1/t} s^{n-k-1} w(s) ds \int\limits_{t^2 s^2}^1 g(u)(u - t^2 s^2)^{k/2-1} du \\
&= \frac{c_{k,n}}{2\Gamma(k/2)} \int\limits_0^{1/t^2} \tau^{(n-k)/2-1} w(\sqrt{\tau}) d\tau \int\limits_\tau^{1/t^2} g(t^2 r)(r-\tau)^{k/2-1} dr \\
&= \frac{c_{k,n}\, t^k}{2} \int\limits_0^{1/t^2} g(t^2 r)(I_{0+}^{k/2} w_1)(r) dr = \frac{c_{k,n}\, t^{k-2}}{2} \int\limits_0^1 g(u) h\left(\frac{u}{t^2}\right) du. \qquad \square
\end{aligned}$$

Remark 3.3. *By Lemma 4.12 of [16], $h \in L^1(\mathbb{R}_+)$ provided*

$$\int\limits_0^\infty \tau^j w_1(\tau) d\tau = 0 \ \ \forall j = 0, 1, \dots, m = \begin{cases} k/2 - 1, & if \ \ k/2 \in \mathbb{N}, \\ [k/2], & if \ \ k/2 \notin \mathbb{N}, \end{cases} \tag{3.4}$$

$$\int\limits_1^\infty \tau^{k/2} |w_1(\tau)| d\tau < \infty. \tag{3.5}$$

Proof of Theorem A. We proceed as in the proof of Theorem 1.1 from [18]. Denote

$$\lambda(s) = \frac{1}{[illegible]}(I_{0+}^{k/2+1} w_1)(s), \quad \Lambda_\varepsilon(\tau) = \frac{c_{k,n}}{[illegible]} \frac{(1-\tau^2)^{1-n/2}}{[illegible]} \lambda\left(\frac{1-\tau^2}{[illegible]}\right), \tag{3.6}$$

$$(\mathcal{L}_\varepsilon f)(x) = \int\limits_{S^n} \Lambda_\varepsilon(x \cdot y) f(y) dy, \tag{3.7}$$

and prove the equality

$$J_\varepsilon Rf(x) \overset{\text{def}}{=} \int\limits_\varepsilon^\infty (\overset{*}{W} Rf)(x,t) \frac{dt}{t^{k+1}} = (\mathcal{L}_\varepsilon f)(x). \tag{3.8}$$

By (3.3),

$$\begin{aligned} J_\varepsilon Rf &= \frac{c_{k,n}}{2} \int\limits_\varepsilon^\infty \frac{dt}{t^3} \int\limits_0^1 g(u) h\left(\frac{u}{t^2}\right) du = \frac{c_{k,n}}{2} \int\limits_0^1 g(u) du \int\limits_\varepsilon^\infty h\left(\frac{u}{t^2}\right) \frac{dt}{t^3} \\ &= \frac{c_{k,n}}{4} \int\limits_0^{1/\varepsilon^2} \lambda(s) g(\varepsilon^2 s) ds. \end{aligned}$$

Using (2.12) from [18] and the definition of g (see Lemma 3.1), we get

$$\begin{aligned} J_\varepsilon Rf &= \frac{c_{k,n}}{4} \int\limits_0^{1/\varepsilon^2} \lambda(s)(1-\varepsilon^2 s)^{-1/2} \mathbb{M}_{\sqrt{1-\varepsilon^2 s}} f ds \\ &= \frac{c_{k,n}}{4\,\varepsilon^2} \int\limits_{-1}^1 (\mathbb{M}_\tau f)(x) \lambda\left(\frac{1-\tau^2}{\varepsilon^2}\right) d\tau \\ &= \frac{c_{k,n}}{4\,\sigma_{n-1}} \int\limits_{S^n} f(y) \left[\frac{(1-(x\cdot y)^2)^{1-n/2}}{\varepsilon^2} \lambda\left(\frac{1-(x\cdot y)^2}{\varepsilon^2}\right)\right] dy, \end{aligned} \tag{3.9}$$

which gives (3.8).

To complete the proof it remains to show that $\mathcal{L}_\varepsilon f \to \alpha f$ as $\varepsilon \to 0$ in the required sense. For this purpose one can use a standard machinery of approximation to the identity. As in ([18], p. 212), for each spherical harmonic Y_j, j even, we have

$$\lim_{\varepsilon \to 0} \mathcal{L}_\varepsilon Y_j \to \alpha Y_j, \qquad \alpha = \frac{c_{k,n}}{4} \int\limits_0^\infty \lambda(s) ds, \tag{3.10}$$

where by Lemma 2.4 from [18],

$$\int\limits_0^\infty \lambda(s) ds = \begin{cases} \Gamma(-k/2) \int\limits_0^\infty s^{k/2} w_1(s) ds & \text{if } k/2 \notin \mathbb{N}, \\ \frac{(-1)^{1+k/2}}{(k/2)!} \int\limits_0^\infty s^{k/2} w_1(s) \log s \; ds & \text{if } k/2 \in \mathbb{N}, \end{cases}$$

provided

$$\int_0^\infty s^j w_1(s)ds = 0, \quad \forall j = 0, 1, \dots, [k/2],$$

$$\int_0^\infty s^\beta |w_1(s)| ds < \infty \quad \text{for some } \beta > k/2.$$

Thus, α in (3.10) has the form (1.11), and the result follows. □

4. Proof of Theorem B

Fix any $t > 0$. By (2.6), $Wf(v,t)$ is well-defined for a.e. $v \in V$. Passing to bispherical coordinates in (1.6), we obtain (cf. (2.8)):

$$Wf(v,t) = \frac{\sigma_{n-k-1}\sigma_k}{t^{n-k}} \int_0^1 (1-\tau^2)^{(k-1)/2} \tau^{n-k-1} w\left(\frac{\tau}{t}\right) M_\tau f(v) d\tau,$$

and therefore

$$\int_\varepsilon^\infty \frac{Wf(v,t)}{t} dt = \sigma_{n-k-1}\sigma_k \int_0^1 (1-\tau^2)^{(k-1)/2} \tau^{n-k-1} M_\tau f(v) d\tau \int_\varepsilon^\infty w\left(\frac{\tau}{t}\right) \frac{dt}{t^{1+n-k}}$$

$$= \sigma_{n-k-1}\sigma_k \int_0^{1/\varepsilon} (1-\varepsilon^2\tau^2)^{(k-1)/2} M_{\varepsilon\tau} f(v) k(\tau) d\tau, \tag{4.1}$$

$k(\tau) = \tau^{-1} \int_0^\tau w(s) s^{n-k-1} ds$. By (1.12),

$$k \in L^1(\mathbb{R}_+) \qquad \text{and} \qquad \int_0^\infty k(\tau) d\tau = \int_0^\infty w(s) s^{n-k-1} \log \frac{1}{s}\, ds$$

(a simple proof of this assertion can be found in ([16], p. 190). Owing to Lemma 2.2, one can pass to the limit in (4.1) in the L^p- and sup-norm as $\varepsilon \to 0$, and the required result follows. □

5. Concluding discussion

It is natural to ask, why do we seek new inversion formulae although so many are available (see Introduction). Below we try to answer this question. It turns out that similar situation arises in *Fractional Calculus*, the branch of analysis which studies fractional integrals and derivatives of functions of one and several variables [16, 20]. The point is that numerous transforms of the Radon type can be included in suitable analytic families of fractional integrals generalizing the classical ones (like those of Riemann-Liouville on the real line or Riesz potentials on $\mathbb{R}^n$) to the case when the set of singularities of the corresponding kernel is a manifold of dimension ≥ 1. This point of view was exhibited in detail in [17]. It enables one to apply methods and ideas of fractional calculus to various problems of integral geometry.

In ”one-dimensional” fractional calculus one can discriminate between the Riemann-Liouville and Marchaud fractional derivatives. In fact, they represent different forms of analytic continuation of the same object, namely, the Riemann-Liouville fractional integral $(I_+^\alpha \psi)(t) = (1/\Gamma(\alpha)) \int_{-\infty}^t \psi(\tau)(t-\tau)^{\alpha-1} d\tau$ (see [20, Section 5] and [16, Section 10] for discussion and further details). The derivative of Marchaud is ”more sensitive” to function spaces (say, L^p, C, Lipschitz or whatever) we are dealing with, while implementation of the Riemann-Liouville fractional derivative for the same purposes may have some restrictions and needs special justification. In many dimensions the method of Marchaud leads to hypersingular integrals which proved to be a powerful tool in function theory for characterization of various spaces of fractional smoothness and inversion of operators of the potential type (see, e.g., the papers by E.M. Stein, P.I. Lizorkin, S.G. Samko, B. Rubin and others mentioned in [16, 20]. Wavelet type representations of fractional integrals and derivatives (in different dimensions) generalize Marchaud's constructions and are more flexilble. The philosophy of such a generalization was developed by B. Rubin [16] who extended this idea to operators of integral geometry [17]. At about the same time continuous wavelet transforms associated to the Radon transform on $\mathbb{R}^n$ were introduced independently by D.L. Donoho and E.J. Candès [3-5], who gave them a new name *ridgelet transforms* (see also N. Murata [15]).

In a sense, formulae (1.2)-(1.4) can be viewed as those of the Riemann-Liouville type, involving additional averaging operators which are inevitable in the integral geometrical set up. Our new formula (1.10) can be treated as that of the Marchaud type (in wavelet interpretation).

Let us examine formulae (1.2)-(1.4) from the point of view of their applicability to functions $f \in L^p(S^n)$ and $f \in C(S^n)$. Of course, all ”doubtful” operations in (1.2)-(1.4) can be treated in the framework of the distribution theory, but this way is too rough and not so interesting. Further, it is known [24] that

$$R^*Rf(x) = c_k \int\limits_{S^n} (1-(x\cdot y)^2)^{(k-n)/2} f(y)dy \stackrel{\text{def}}{=} A_k f(x), \qquad c_k = \text{const.} \tag{5.1}$$

This is an operator of the potential type. In accordance with the Sobolev embedding theorem the smoothness of $A_k f(x)$ for $f \in L^p(S^n)$ depends on interrelation between k, n and p. The latter means that differentiation in (1.2) must be treated accordingly. A similar problem arises in Strichartz' formula (1.4) by taking into account that $E_{-1}R^*Rf$ is a Riesz potential $I^k\tilde{f}$ of a certain continuation $\tilde{f}$ of f onto $\mathbb{R}^{n+1}$ by homogeneity (see [24, Theorem 4.7]). Thus, in both cases a special investigation concerning interpretation of differentiation in (1.2) and (1.4) is needed (cf., e.g., [14. p. 242], [22, Chapter VIII]). In (1.3) we have a similar problem with differentiation. Moreover, another operation

$$[\dots] \to [\dots]|_{u=1} \tag{5.2}$$

also needs justification. Before setting $u=1$, (1.3) gives an average $\mathbb{M}_u f(x)$ of f over the geodesic sphere centered at x of radius $\cos^{-1}u$ (see (2.9)). If $f \in C(S^n)$, then $\mathbb{M}_u f(x)$ is a continuous function of $u \in [-1,1]$ for each $x \in S^n$. Hence (5.2) does not cause any trouble in this case. If $f \in L^p(S^n)$, then, in general, one cannot set $u=1$ directly, and we only have $\lim\limits_{u\to 1} \mathbb{M}_u f(x) = f(x)$ in the L^p-norm. A similar equality in the almost everywhere sense seems to be an open problem related to L^p-boundedness of the maximal operator $f \to \sup\limits_u |f(x)|$. An analogue of this operator in the context of $\mathbb{R}^n$ is bounded if and only if $p > n/(n-1)$ [23, p. 471]. By taking into account related results of C.D. Sogge and his collaborators, it is natural to conjecture the validity of the same statement for S^n.

Resuming this discussion, we note that while (1.2)-(1.4) need additional investigation, our inversion formula (1.10) is completely justified and coherent with the L^p-setting of the problem. Moreover, (1.10) has many degrees of freedom because one can choose any wavelet function w he likes. This is especially important in numerical calculations where it is desirable to have w smooth and well localized. Note also that (1.10) has the same form as inversion formulas for many other Radon type transforms [17] written in the wavelet language.

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