Singular integrals, generated by spherical measures

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Abstract. In this paper we study the $L^p$-mapping properties of the Calderón-Zygmund type singular integral operator $T_\nu f(x) = \int_0^\infty dr/r \int_{\Sigma_{n-1}} f(x - r\theta)d\nu(\theta)$, depending on a finite Borel measure $\nu$. In particular it is shown that the conditions $\nu(\Sigma_{n-1}) = 0$, $\sup |\xi| \log (1/|\theta \cdot \xi|)d|\nu|(\theta) < \infty$ imply the $L^p$-boundedness of $T_\nu$, $1 < p < \infty$ provided $n > 2$, and $\nu$ is zonal.

1. Introduction.

Let $\Sigma_{n-1}$ denote the unit sphere in $\mathbb{R}^n$, and let $\Omega \in L^1(\Sigma_{n-1})$, $\int_{\Sigma_{n-1}} \Omega(\theta)d\theta = 0$. Consider the Calderón-Zygmund singular integral operator

\begin{equation}
(T_\Omega f)(x) = \lim_{\rho \to \infty} \frac{1}{|\rho|} \int_{|x - y| < \rho} f(x - y) \frac{\Omega(y/|y|)}{|y|^n} dy,
\end{equation}

arising in a variety of problems (we refer the reader to the books [18], [19], [8], [4], [17] and the survey article [6] for more background information).

It is well-known [1], that if $\Omega \in L^1(\Sigma_{n-1})$ is odd, then the limit in (1.1) exists in the $L^p$-norm and a.e., for all $f \in L^p(\mathbb{R}^n)$, $1 < p < \infty$. This is a consequence of the corresponding one-dimensional result and the method of rotations. The main difficulty is connected with the case of $\Omega$ even. The following result of W. Connett [2], F. Ricci and G. Weiss [11], is well-known (see also [21], [6], [15]).

Theorem 1.1. Let $f \in L^p(\mathbb{R}^n), 1 < p < \infty$. If $\Omega$ belongs to the Hardy space $H^1(\Sigma_{n-1})$, and $\int_{\Sigma_{n-1}} \Omega(\theta)d\theta = 0$, then

\begin{equation}
\| \sup_{0 < \rho < \infty} |T_\Omega^{(\rho)} f| \|_p \leq c_p \|f\|_p,
\end{equation}

and the limit (1.1) exists in the $L^p$-norm and a.e.

We also mention the following $L^2$-result (cf. [18], p. 40).

Theorem 1.2. If

\begin{equation}
\int_{\Sigma_{n-1}} \Omega(\theta)d\theta = 0 \quad \text{and} \quad \sup_{|\xi| = 1} \int_{\Sigma_{n-1}} |\Omega(\theta)| \log \frac{1}{|\theta \cdot \xi|} d\theta < \infty,
\end{equation}

then $T_\Omega$ is bounded from $L^2(\mathbb{R}^n)$ into itself.

L. Grafakos and A. Stefanov [7] considered the class of functions $\Omega(\theta)$ satisfying the following conditions:

\begin{equation}
\int_{\Sigma_{n-1}} \Omega(\theta)d\theta = 0, \quad \sup_{|\xi| = 1} \int_{\Sigma_{n-1}} |\Omega(\theta)| \left( \log \frac{1}{|\theta \cdot \xi|} \right)^{1+\alpha} d\theta < \infty, \quad \alpha > 0.
\end{equation}

They showed that this class is different from $H^1(\Sigma_{n-1})$ and proved the following theorem.
**Theorem 1.3** ([7]). If $\Omega$ satisfies (1.4), then $T_\Omega$ extends to a bounded operator from $L^p$ into itself for $2 - \alpha/(1 + \alpha) < p < 2 + \alpha$. If, moreover, $\alpha > 1$, then (1.2) holds for $2 - (2 + 2\alpha)/(1 + 2\alpha) < p < 2 + (2\alpha - 2)/3$.

The method of the proof of Theorem 1.3 is based on ideas, which were developed by J. Duoandikoetxea and J. L. Rubio de Francia [3]. The following questions were posed in [7, pp. 456, 457]:

**Question 1.** Are the ranges of indices in Theorem 1.3 sharp?

**Question 2.** Does the conditions (1.3) imply the $L^p$-boundedness of $T_\Omega$ for some $p \neq 2$?

In this paper we extend the aforementioned ranges of indices and show that (1.3) implies the $L^p$-boundedness of $T_\Omega$ for all $p \in (1, \infty)$ in the case $n > 2$ provided that $\Omega$ is zonal (i.e. invariant under all rotations about the $x_n$-axis). We also consider a generalization of $T_\Omega$ with $\Omega$ replaced by a finite Borel measure on $\Sigma_{n-1}$. More precisely, let $M(\Sigma_{n-1})$ be a space of all such measures. Given $\nu \in M(\Sigma_{n-1})$, consider the singular integral operator

\[
(T_\nu f)(x) = \lim_{\varepsilon \to 0} (T_{\nu, \varepsilon} f)(x) = \lim_{\rho \to \infty} \lim_{\varepsilon \to 0} \frac{d\nu}{\varepsilon} \int_{\Sigma_{n-1}} f(x - r\theta) d\nu(\theta).
\]

If $\nu$ is absolutely continuous with respect to the Lebesgue measure $d\theta$ on $\Sigma_{n-1}$, i.e.

\[
d\nu(\theta) = \Omega(\theta) d\theta,
\]

$\Omega \in L^1(\Sigma_{n-1})$, then (1.5) coincides with (1.1).

Let us state our main results. The following theorems are related to Question 1.

**Theorem A.** Let

\[
\nu(\Sigma_{n-1}) = 0, \quad \sup_{|\xi|=1} \int_{\Sigma_{n-1}} \left( \log \frac{1}{|\theta \cdot \xi|} \right)^{1+\alpha} d|\nu|(\theta) < \infty \quad \text{for some} \quad \alpha > 0.
\]

Then the operator $T_\nu$, initially defined by (1.5) on functions $f \in C_\infty^\infty(\mathbb{R}^n)$, extends to a linear bounded operator from $L^p$ into itself provided

\[
\left| \frac{1}{2} - \frac{1}{p} \right| < \frac{\alpha}{2(1 + \alpha)}.
\]
Theorem B. Suppose that \( f \in L^p(\mathbb{R}^n) \), and \( \nu \in M(\Sigma_{n-1}) \) satisfies (1.6) for some \( \alpha > 1 \). Then
\[
\| \sup_{0 < \varepsilon < \rho < \infty} |T^\varepsilon \rho f| \|_p \leq c_p \| f \|_p, \tag{1.8}
\]
provided
\[
\left| \frac{1}{2} - \frac{1}{p} \right| < \frac{\alpha - 1}{2\alpha}, \tag{1.9}
\]
and the limit in (1.5) exists in the \( L^p \)-norm, and in the a.e. sense.

As in [7] and in many other papers, related to singular integral operators, we employ the ideas developed by J. Duoandikoetxea and J. L. Rubio de Francia in [3]. The possibility of extending the bounds for \( p \) is based on the use of the method of rotations, instead of a “bootstrap” argument (cf. [17], p. 463), which was used in [7], p. 460.

Our next result concerns Question 2. Let \( M_z(\Sigma_{n-1}) \) be the subspace of \( M(\Sigma_{n-1}) \), consisting of zeroal measures.

Theorem C. Suppose that \( \nu \in M_z(\Sigma_{n-1}) \), \( \nu(\Sigma_{n-1}) = 0 \), \( n > 2 \).

(a) If
\[
\int_{\Sigma_{n-1}} \log \frac{1}{|\theta_n| \sqrt{1 - \theta_n^2}} d|\nu|(\theta) < \infty, \tag{1.10}
\]
then \( T_\nu \) extends to a bounded operator from \( L^p \) into itself for all \( p \in (1, \infty) \).

(b) Let \( f \in L^p(\mathbb{R}^n), 1 < p < \infty \). If
\[
\int_{\Sigma_{n-1}} |\theta_n|^{-\beta} (1 - \theta_n^2)^{-\beta/2} d|\nu|(\theta) < \infty \quad \text{for some } \beta \in (0, 1/2), \tag{1.11}
\]
then (1.8) holds, and the limit in (1.5) exists in the \( L^p \)-norm and in the a.e. sense.

The proof of part (a) of this theorem employs recent results of D. K. Watson [20].

Corollary 1.4 (cf. Theorem 1.2). Let \( n > 2 \), and let \( \nu \in M_z(\Sigma_{n-1}) \) satisfy (1.6) with \( \alpha = 0 \). Then \( T_\nu \) extends to a bounded operator from \( L^p \) into itself for all \( p \in (1, \infty) \).

Corollary 1.5. Let \( n > 2 \). There is an even function \( \Omega \notin H^1(\Sigma_{n-1}) \) which satisfies (1.3) and does not satisfy (1.4) for any \( \alpha > 0 \), but, nevertheless, the relevant operator \( T_\Omega \) extends to bounded operators from \( L^p \) into itself for all \( p \in (1, \infty) \).

The above corollary shows that the ranges of indices in (1.7) are also not sharp.
Corollary 1.6. Let $n > 2$. There is an even function $\Omega \notin H^1(\Sigma_{n-1})$, which satisfies (1.4) for all $\alpha > 0$.

This result was proved in [7] for $n = 2$. But the proof, given there, was fairly complicated. We show that (for $n > 2$) examples of functions indicated in Corollary 1.6, can be easily obtained from Theorem C and geometric properties of the Hardy spaces $H^1(\Sigma_{n-1})$ and $H^1(\mathbb{R}^n)$.

We do not know if the results of Theorem C and Corollaries 1.4, 1.5 are true in the case $n = 2$. Another open problem is whether Corollary 1.4 holds for non-zonal $\nu$ if $p \neq 2$.

The following observation related to Theorems 1.1 and 1.2 is also of interest. Namely, the second condition in (1.3) may fail, but nevertheless, $T_\Omega$ is bounded on $L^p$ for all $1 < p < \infty$. More precisely, the following statement holds.

**Proposition 1.7.** There is an even function $\Omega \in H^1(\Sigma_{n-1})$ such that $\int_{\Sigma_{n-1}} \Omega(\theta) d\theta = 0$ and

$$
\sup_{|\xi| = 1} \int_{\Sigma_{n-1}} |\Omega(\theta)| \log \frac{1}{|\xi \cdot \theta|} d\theta = \infty.
$$

The paper is organized as follows. In section 2 we prove Theorem A. Sections 3 and 4 are devoted to the proof of Theorem B. The proof of Theorem C and Corollaries 1.4-1.6 is given in section 5. In section 6 we prove Proposition 1.7 and in section 7 give examples of non-zonal singular measures, satisfying (1.6) for all $\alpha > 0$.

**Notation.** Let $\Sigma_{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$, $\sigma_{n-1} = |\Sigma_{n-1}| = 2\pi^{n/2}/\Gamma(n/2)$; $M(X)$ denotes the space of all finite Borel measures on a measure space $X$; $|\mu|$ designates the total variation of $\mu \in M(X)$. The notation $L^p(X)$ is standard; $C_0(\mathbb{R}^n)$ denotes the space of continuous on $\mathbb{R}^n$ functions, tending to zero at infinity; $C^\infty(\mathbb{R}^n)$ is the space of infinitely differentiable on $\mathbb{R}^n$ functions, having a compact support. We define the Fourier transform of $\mu \in M(\mathbb{R}^n)$ by $\hat{\mu}(\xi) = \int_{\mathbb{R}^n} e^{-2\pi i x \cdot \xi} d\mu(x)$. The group of rotations leaving the $x_n$-axis fixed will be denoted by $SO(n-1)$; $e_n = (0, \ldots, 0, 1)$. A measure $\nu \in M(\Sigma_{n-1})$ is called zonal if $\int_{\Sigma_{n-1}} f(\gamma \vartheta) d\nu(\vartheta) = \int_{\Sigma_{n-1}} f(\vartheta) d\nu(\vartheta)$ for each $\gamma \in SO(n-1)$ and each $f \in L^1(\Sigma_{n-1}, d\nu)$. The set of all zonal measures on $\Sigma_{n-1}$ is denoted by $M_z(\Sigma_{n-1})$. The letter $c$ designates a constant, not necessarily the same at each occurrence.
2. Proof of Theorem A.

We begin by proving some auxiliary statements. Following [3], let \( \{ \psi_j \}_{j \in \mathbb{Z}} \) be a smooth partition of the unity on \((0, \infty)\) so that
\[
\begin{align*}
&\text{a)} \quad \psi_j \in C^1(\mathbb{R}_+), \quad 0 \leq \psi_j \leq 1, \quad \sum_{j \in \mathbb{Z}} \psi_j^2(t) = 1, \\
&\text{b)} \quad \text{supp}(\psi_0) \subseteq \{ t \in \mathbb{R} : 1/2 \leq t \leq 2 \}, \quad \psi_j(t) = \psi_0(2^j t), \\
&\text{c)} \quad \psi_0(t) \equiv 1 \quad \forall t \in [1, 3/2], \quad |\psi_j'(t)| \leq c/t.
\end{align*}
\]
Suppose also that \( \sigma_k(\in M(\mathbb{R}^n)) \), \( k \in \mathbb{Z} \), is a sequence of measures such that
\[
\|\sigma_k\| \leq 1, \quad \text{supp} \ \sigma_k \subseteq \{ x \in \mathbb{R}^n : 2^k \leq |x| \leq 2^{k+1} \},
\]
and
\[
|\hat{\sigma}_k(\xi)| \leq \begin{cases} 
\frac{c |2^k \xi|}{2^k} & \text{if } |2^k \xi| \leq 2, \\
\frac{c}{\log^{-1-\alpha}} |2^k \xi| & \text{if } |2^k \xi| > 2, \quad \alpha > 0.
\end{cases}
\]
For \( f \in C_c^\infty(\mathbb{R}^n) \), we define
\[
Tf(x) = \sum_{k \in \mathbb{Z}} (\sigma_k * f)(x),
\]
and
\[
(S_j f)\hat{}(\xi) = \hat{f}(\xi) \psi_j(|\xi|), \quad (T_j f)(x) = \sum_{k \in \mathbb{Z}} S_{j+k}(\sigma_k * S_{j+k} f)(x), \quad j \in \mathbb{Z}.
\]

**Lemma 2.1.** Let \( f \in C_c^\infty(\mathbb{R}^n) \), and
\[
\|\sup_{k \in \mathbb{Z}} (|\sigma_k| * |f|)\|_s \leq c \|f\|_s \quad \forall s \in (1, \infty).
\]
Then
\[
\|T_j f\|_q \leq c \|f\|_q \quad \text{for all } q \in (1, \infty).
\]
If, moreover, \( \sigma_k, \ k \in \mathbb{Z}, \) satisfy (2.2), then for all \( \lambda \in [0, 1] \),
\[
\|T_j f\|_p \leq c (1 + |j|)^{-(1+\lambda)} \|f\|_p,
\]
provided that \( \lambda/2 < 1/p < 1 - \lambda/2 \) if \( 0 \leq \lambda < 1 \), and \( p = 2 \) if \( \lambda = 1 \). The constant \( c \) in (2.6) and (2.7) is independent of \( j \).

The estimate (2.7) for the smaller range of \( p \)'s was proved in [7], p. 460 (it was a consequence of a “bootstrap argument” and the assumption that (2.5) holds for \( s = 2 \),
But the point is that in the studying of the operator (1.5), we can always assume that the maximal estimate, corresponding to (2.5) holds for a full range of $s$ (see (2.13)).

**Proof of Lemma 2.1.** The estimate (2.6) was established in [3], p. 545, and we recall its proof for convenience of the reader. We have

$$
\|T_j f\|_q \leq c \left( \sum_{k \in \mathbb{Z}} |S_{j+k}(\sigma_k * S_{j+k} f)|^2 \right)^{1/2} \|q \leq c \left( \sum_{k \in \mathbb{Z}} |\sigma_k * S_{j+k} f|^2 \right)^{1/2} \leq c \|f\|_q.
$$

Here (1) and (4) follow from the Littlewood-Paley theory [17], p. 267; (2) is a special case of the more general estimate (4) from [13]; (3) holds according to the lemma on p. 544 from [3]. Furthermore, by Plancherel’s theorem,

$$
\|T_j f\|_2 \leq \sum_{k \in \mathbb{Z}} \|\hat{\sigma}_k \hat{\psi}_{j+k} f\|_2 \leq \sum_{k \in \mathbb{Z}} \left( \int_{\text{supp } \hat{\psi}_{j+k}} |\hat{\sigma}_k(\xi)|^2 |\hat{f}(\xi)|^2 d\xi \right)^{1/2}.
$$

Owing to (2.2), this gives (2.7) for $\lambda = 1$, $p = 2$ (cf. formula (11) from [7]). For $\lambda = 0$, (2.7) coincides with (2.6). The result for $0 < \lambda < 1$ follows by interpolation.

**Lemma 2.2.** Suppose that $\sigma_k \in M(\mathbb{R}^n)$, $k \in \mathbb{Z}$, satisfy (2.1), (2.2), and (2.5). Then (2.3) extends to a linear bounded operator on $L^p(\mathbb{R}^n)$ provided $1/2 - 1/p < \alpha (2(1 + \alpha))^{-1}$.

**Proof.** As in [3, p. 545], for $f \in C_c^\infty(\mathbb{R}^n)$ we have:

$$
(2.8) \quad T f = \sum_{k \in \mathbb{Z}} \sigma_k * f = \sum_{j \in \mathbb{Z}} T_j f,
$$

and (2.7) yields $\|T f\|_p \leq c \|f\|_p \sum_j (1 + |j|)^{-(1+\alpha)\lambda}$, $\lambda/2 < 1/p < 1 - \lambda/2$. Assuming $\lambda > (1 + \alpha)^{-1}$, we obtain the required result.

Now we pass to the operator $T_\nu$ from (1.5). One can write $T_\nu f = \sum_{k \in \mathbb{Z}} \omega_k * f$, where $\omega_k \in M(\mathbb{R}^n)$ are defined by

$$
(2.9) \quad \int_{\mathbb{R}^n} g(y) d\omega_k(y) = c_{\nu} \int_{\Sigma_{n-1}} ^{(2k+1)} \int \frac{dr}{\tau} g(r\theta) d\nu(\theta), \quad c_{\nu} = (\nu(\Sigma_{n-1}) \log 2)^{-1},
$$

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\( g \in C_0(\mathbb{R}^n), \) and \( \nu \) satisfies (1.6). Denote by \( \mathcal{B}(\mathbb{R}^n) \) the \( \sigma \)-algebra of all Borel measurable sets in \( \mathbb{R}^n \). As usual [14, p. 116],

\[
(2.10) \quad |\omega_k|(E) = \sup_{\{A_i\}} \sum_{i=1}^{\infty} |\omega_k(A_i)|, \quad E \in \mathcal{B}(\mathbb{R}^n),
\]

where the supremum is taken over all partitions of \( E \) by \( A_i \in \mathcal{B}(\mathbb{R}^n) \).

**Lemma 2.3.** Let \( E \in \mathcal{B}(\mathbb{R}^n) \), \( 1_E(x) = 1 \) for \( x \in E \) and \( 1_E(x) = 0 \) otherwise. Then

\[
(2.11) \quad |\omega_k|(E) \leq c_\nu \int_{2^k}^{2^{k+1}} \frac{dr}{r} \int_{\Sigma_{n-1}} 1_E(r\theta)d|\nu|(\theta) \leq 1.
\]

Furthermore, for \( f \in L^s(\mathbb{R}^n), \) \( 1 < s < \infty, \) the following relations hold:

\[
(2.12) \quad (|\omega_k| * |f|)(x) \leq c_\nu \int_{\Sigma_{n-1}} d|\nu|(\theta) \int_{2^k}^{2^{k+1}} |f(x - r\theta)| \frac{dr}{r},
\]

\[
(2.13) \quad \left\| \sup_{k \in \mathbb{Z}} (|\omega_k| * |f|) \right\|_s \leq c \left\| f \right\|_s.
\]

**Proof.** The first relation follows by (2.9), (2.10):

\[
|\omega_k|(E) \leq c_\nu \sup_{\{A_i\}} \sum_{i} \int_{2^k}^{2^{k+1}} \frac{dr}{r} \int_{\Sigma_{n-1}} 1_{A_i}(r\theta)d|\nu|(\theta) \leq c_\nu \int_{2^k}^{2^{k+1}} \frac{dr}{r} \int_{\Sigma_{n-1}} \left( \sup_{\{A_i\}} \sum_{i} 1_{A_i}(r\theta) \right)d|\nu|(\theta) = c_\nu \int_{2^k}^{2^{k+1}} \frac{dr}{r} \int_{\Sigma_{n-1}} 1_E(r\theta)d|\nu|(\theta) \leq 1.
\]

Furthermore, if \( f \in C_0(\mathbb{R}^n), \) \( f^x(y) = |f(x - y)|, \) then by Theorem 1.17 from [14, p. 15] there is a sequence \( \{S^x_m(y)\}_{m=1}^{\infty} \) of simple functions, such that \( 0 \leq S^x_1 \leq S^x_2 \leq \ldots \leq S^x_m \leq \ldots \leq |f^x| \) and \( S^x_m(y) \to |f^x(y)| \) for each \( x \) and \( y. \) Hence

\[
(|\omega_k| * |f|)(x) = (|\omega_k|, \lim_{m \to \infty} S^x_m) = \lim_{m \to \infty} (|\omega_k|, S^x_m) \leq \lim_{m \to \infty} c_\nu \int_{2^k}^{2^{k+1}} \frac{dr}{r} \int_{\Sigma_{n-1}} S^x_m(r\theta)d|\nu|(\theta) = c_\nu \int_{2^k}^{2^{k+1}} \frac{dr}{r} \int_{\Sigma_{n-1}} |f(x - r\theta)|d|\nu|(\theta).
\]
In the general case $f \in L^s(\mathbb{R}^n)$, (2.12) then follows by the limiting argument from its validity for any convolution $(|f| * g_t)(x)$, $g_t(x) = t^{-n} g(x/t)$, $g \in C^\infty_c(\mathbb{R}^n)$, $g \geq 0$.

To prove (2.13), we denote by

$$(M_\theta f)(x) = \sup_{R > 0} \frac{1}{R} \int_0^R |f(x - r\theta)| dr$$

the one-dimensional Hardy-Littlewood maximal operator in direction $\theta \in \Sigma_{n-1}$. By the method of rotations

$$\|M_\theta f\|_s \leq c \|f\|_s, \quad s > 1,$$

c being independent of $\theta$. Then

$$([\omega_k * |f|](x)) \leq 2c \nu \int_{\Sigma_{n-1}} \left( \left\lfloor \frac{2^{k+1}}{2^k} \right\rfloor \int |f(x - r\theta)| dr \right) |d\nu|(\theta) \leq 2c \nu \int_{\Sigma_{n-1}} (M_\theta f)(x) d|\nu|(\theta),$$

and the result follows by (2.14).

**Lemma 2.4.** Let $\nu \in M(\Sigma_{n-1})$ satisfy (1.6). Then there is a constant $c = c(\alpha, \nu) > 0$ such that for all $k \in \mathbb{Z}$,

$$|\hat{\omega}_k(\xi)| \leq \begin{cases} c \left| 2^k \xi \right| & \text{if } \left| 2^k \xi \right| \leq 2, \\ c \log^{-1/\alpha} \left| 2^k \xi \right| & \text{if } \left| 2^k \xi \right| > 2. \end{cases}$$

This statement resembles the estimate (10) from [7]; see also [3, p. 550]. For convenience of the reader we prove (2.15) by completing some details, which were omitted in [7].

**Proof.** Since $\hat{\omega}_k(\xi) = \hat{\omega}_0(2^k \xi)$, it suffices to consider $k = 0$. The inequality

$$|\hat{\omega}_0(\xi)| = \left| c \nu \int_1^2 \frac{dr}{r} \int_{\Sigma_{n-1}} e^{-2\pi i r \theta \cdot \xi} d\nu(\theta)\right| \leq c |\xi|, \quad |\xi| \leq 2,$$

is clear because $\nu(\Sigma_{n-1}) = 0$. The inequality $|\hat{\omega}_0(\xi)| \leq c (\log |\xi|)^{-1-\alpha}$, $|\xi| > 2$, follows by (1.6) from the estimate

$$A \overset{\text{def}}{=} \left| \int_1^2 e^{-2\pi i r \theta \cdot \xi} \frac{dr}{r} \right| \leq c \left( \frac{b}{a} \right) \gamma, \quad a = \log |\xi|, \quad b = \log \frac{3/2}{|\theta \cdot \xi|^r},$$

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$c = c(\gamma) = \text{const.}$ The latter holds for all $\gamma \geq 0$ (in our case $\gamma = 1 + \alpha$), $\theta \cdot \xi \neq 0$, $\xi' = \xi/|\xi|$.

Let us prove (2.16). Integration by parts yields

$$A = \left| \int_1^2 \frac{d(e^{-2\pi ir \theta \cdot \xi})}{2\pi ir \theta \cdot \xi} \right| \leq \frac{1}{\pi |\theta \cdot \xi|} \leq \frac{3/2}{|\theta \cdot \xi|} = e^{b-a}.$$ 

Note also that $b > \log(3/2) > 1/4$, i.e., $1 < 4b$. If $a - b \geq 1$, then $a/(a - b) \leq 1 + b \leq 5b$, and therefore

$$A \leq \frac{(a - b)^{\gamma} e^{-(a-b)}}{(a-b)^{\gamma}} \leq c \frac{\gamma}{(a-b)^{\gamma}} \leq c \gamma \left( \frac{5b}{a} \right)^{\gamma}.$$ 

If $a - b < 1$, then $a/b \leq (b+1)/b = 1 + 1/b < 5$ and we get $A \leq \int_1^2 dr/r < 1 < (5b/a)^{\gamma}$. \Lambda

It remains to note that Theorem A is a consequence of Lemmas 2.2 - 2.4 (put $\sigma_k = \omega_k$, where $\omega_k$ are defined by (2.9)).

3. Auxiliary statements.

Suppose that $T$ is the operator (2,3), $\Phi(x)$ is a Schwartz function and $\Phi_j(x) = 2^{-jn}\Phi(2^{-j}x)$.

**Lemma 3.1.** Let $f \in C_c^\infty(\mathbb{R}^n)$. If $\|Tf\|_p \leq c \|f\|_p$, then

$$\| \sup_{j \in \mathbb{Z}} \| \Phi_j \sum_{k=j}^\infty \sigma_k \ast f \|_p \leq c \|f\|_p. \quad (3.1)$$

The proof of this statement is given in [3], p. 548, and employs the estimate

$$\left| \sum_{k=-\infty}^{j-1} (\sigma_k \ast \Phi_j)(y) \right| = \left| (\Phi_j \ast \sum_{k=-\infty}^{j-1} \sigma_k)(y) \right| \leq c \psi_j(y), \quad (3.2)$$

$\psi_j(y) = 2^{-jn}/(1 + |2^{-j}y|)^{n+1}$. Since this estimate will be used below and the proof of it was skipped in [3], we complete the details. For $x \in \text{supp } \sigma_k$, $y \in \mathbb{R}^n$, define $h_{x,y}(t) = \Phi_j(y - tx)$, $t \in [0, 1]$. By (2.2), $\sigma_k(0) = \sigma_k(R^n) = 0$. Hence the left hand side of (3.2) does not exceed

$$\sum_{k=-\infty}^{j-1} \int_{\mathbb{R}^n} |h'_{x,y}(\eta)| d|\sigma_k|(x) \leq \sum_{k=-\infty}^{j-1} \int_{\mathbb{R}^n} \sum_{l=1}^n \left| \frac{\partial \Phi_j}{\partial \xi_l}(y - \eta x) \right| |x_l| d|\sigma_k|(x),$$

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where $\eta = \eta(x, y) \in [0, 1]$, $\xi_t = y_t - \eta_t x$. The above expression is estimated by

$$
2^{-j(n+1)} \sum_{k=-\infty}^{j-1} \int_{2^k < |x| < 2^{k+1}} \frac{1 + |x|^2}{1 + y^2} |x| d|\sigma_k|(x) \leq c \sum_{k=-\infty}^{j-1} 2^{-j(n+1)} \int_{2^k < |x| < 2^{k+1}} \frac{1 + |x|^2}{1 + y^2} |x| d|\sigma_k|(x) \leq c 2^{n+1} \sum_{k=-\infty}^{j-1} 2^{-j+k+1} \psi_j(y),
$$

which gives (3.2).

We recall Cotlar’s lemma, which will be used below.

**Lemma 3.2** ([17], p. 280). Suppose that $\{Q_t\}$ is a finite collection of bounded operators on $L^2(\mathbb{R}^n)$. Assume that we are given a sequence of positive constants $\{\gamma(t)\}_{t \in \mathbb{Z}}$ with

$$
A = \sum_{t \in \mathbb{Z}} \gamma(t) < \infty,
$$

and

$$
\|Q_t^*Q_k\| \leq [\gamma(i-k)]^2, \quad \|Q_tQ_k^*\| \leq [\gamma(i-k)]^2;
$$

here $\| \cdot \|$ denotes the operator norm on $L^2$. Then the operator $Q = \sum_t Q_t$ satisfies $\|Q\| \leq A$.

For $\Phi_k(x)$ as in Lemma 3.1 and for $\omega_k$ defined by (2.9), we get

$$
Q_j f = \sup_{k \in \mathbb{Z}} |f_{j,k}|, \quad f_{j,k} = \omega_{j+k} * f - \Phi_k * \omega_{j+k} * f.
$$

**Lemma 3.3.** Let $j \geq 0$. There is a constant $c > 0$, independent of $j$, with the following properties.

(i) If $f \in L^q(\mathbb{R}^n)$, $1 < q < \infty$, then

$$
\|Q_j f\|_q \leq c \|f\|_q.
$$

(ii) If $\Phi$ is a radial function, such that $|\hat{\Phi}(\xi)| \leq 1$, $\hat{\Phi}(\xi) = 1$ for $|\xi| \leq 2$ and $\hat{\Phi}(\xi) = 0$ for $|\xi| > 3$, and $f \in L^2(\mathbb{R}^n)$, then

$$
\|Q_j f\|_2 \leq c (1 + j)^{-\alpha} \|f\|_2,
$$

where $\eta = \eta(x, y) \in [0, 1]$, $\xi_t = y_t - \eta_t x$. The above expression is estimated by

$$
2^{-j(n+1)} \sum_{k=-\infty}^{j-1} \int_{2^k < |x| < 2^{k+1}} \frac{1 + |x|^2}{1 + y^2} |x| d|\sigma_k|(x) \leq c \sum_{k=-\infty}^{j-1} 2^{-j(n+1)} \int_{2^k < |x| < 2^{k+1}} \frac{1 + |x|^2}{1 + y^2} |x| d|\sigma_k|(x) \leq c 2^{n+1} \sum_{k=-\infty}^{j-1} 2^{-j+k+1} \psi_j(y),
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$$
\|Q_j f\|_2 \leq c (1 + j)^{-\alpha} \|f\|_2,
$$
provided that \( \nu \) satisfies (1.6) with \( \alpha > 1 \).

**Proof.** (i) We have \( Q_j f \leq \sup_{k \in \mathbb{Z}} |\omega_{j+k} * f| + c M [\sup_{k \in \mathbb{Z}} |\omega_{j+k} * f|] \), where \( M \) is the Hardy-Littlewood maximal operator. Hence (3.6) follows by Lemma 2.3.

(ii) Since \( (\sup_{k} |f_{j,k}|)^2 \leq \sup_{k} |f_{j,k}|^2 \leq \sum_{k} |f_{j,k}|^2 \), then

\[
\|Q_j f\|_2^2 \leq \left\| \sum_{k \in \mathbb{Z}} |f_{j,k}(x)|^2 \right\|_2 = \lim_{N \to \infty} \sum_{|k| \leq N} \int_{\mathbb{R}^n} |f_{j,k}(x)|^2 dx.
\]

It suffices to show that for an arbitrary \( N \in \mathbb{N} \),

\[
(3.8) \quad \sum_{|k| \leq N} \int_{\mathbb{R}^n} |f_{j,k}(x)|^2 dx \leq c (1 + j)^{-2\alpha} \|f\|_2^2,
\]

where \( c \) is independent of \( j \) and \( N \). To prove (3.8) we make use of Lemma 3.2. Let \( \{r_t(t)\}_{t=1}^\infty \) be an orthonormal system of the Rademacher functions in \( L^2[0,1] \) so that \( r_t(t) = \text{sgn} \sin 2^t t \pi \),

\[
\sum_{|k| \leq N} \int_{\mathbb{R}^n} |f_{j,k}(x)|^2 dx = \int_0^1 dt \int_{\mathbb{R}^n} \left| \sum_{|k| \leq N} r_{k+N+1}(t) f_{j,k}(x) \right|^2 dx,
\]

(cf. [22], p. 176, 180). Fix \( N \in \mathbb{N} \), \( t \in [0,1] \), and set \( Q_{k,N}^j f = r_{k+N+1}(t) f_{j,k} \). We claim that,

\[
(3.9) \quad \|Q_{k,N}^j \|_{2 \to 2} \leq \left| \gamma_j(k-i) \right|^2, \quad \gamma_j(k-i) = c \frac{(1+j)^{-(1+\alpha)/2}}{(1+j+|k-i|)^{(1+\alpha)/2}},
\]

where \( c \) is independent of \( r_{k+N+1} \) and \( N \) (the same estimate holds for \( Q_{k,N}^j (Q_{k,N}^j)^* \)).

Suppose for a moment, that (3.9) is true. Then

\[
\sum_{|\ell| \leq L} \gamma_j(\ell) = c (1+j)^{-(1+\alpha)/2} \sum_{|\ell| \leq L}(1+j+|\ell|)^{-(1+\alpha)/2} \leq 2c (1+j)^{-(1+\alpha)/2} \int_0^\infty (1+j+t)^{-(1+\alpha)/2} dt = c (1+j)^{-\alpha}, \quad \alpha > 1,
\]

and Lemma 3.2 yields

\[
\| \sum_{|k| \leq N} Q_{k,N}^j f \|_2 = \| \sum_{|k| \leq N} r_{k+N+1}(t) f_{j,k} \|_2 \leq c (1+j)^{-\alpha} \|f\|_2,
\]

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c being independent of \( r_{k+n+1}(t) \) and \( N \). This implies (3.8).

Let us prove (3.9). By Plancherel's Theorem and the definition of \( \Phi_k \) \( (\Phi_k(\xi) = 1 \) for \( 2^\ell|\xi| \leq 2 \),

\[
\|(Q_{i,N}^j)^* Q_{i,N}^j f\|_2^2 \leq \int_{\mathbb{R}^n} |1 - \Phi_k(\xi)|^2 |1 - \Phi_i(\xi)|^2 |\hat{\omega}_{j+k}^j(\xi)\hat{\omega}_{j+i}^i(\xi)|^2 |\hat{f}(\xi)|^2 d\xi \leq \int_{|\xi| > 2^{1 - \min(i,k)}} |\hat{\omega}_{j+k}^j(\xi)\hat{\omega}_{j+i}^i(\xi)|^2 |\hat{f}(\xi)|^2 d\xi.
\]

By Lemma 2.4 the last integral does not exceed

\[
\int_{|\xi| > 2^{1 - \min(i,k)}} |\hat{f}(\xi)|^2 |\log(2^j+k|\xi|)\log(2^j+i|\xi|)|^{-2-2\alpha} d\xi \leq c^2 a_{i,k}^j \|\hat{f}\|_2^2,
\]

where \( a_{i,k}^j = [(j+k+1-\min(i,k))(j+i+1-\min(i,k))]^{-2-2\alpha} = [(j+1)(j+1+|k-i|)]^{-2-2\alpha} \), and (3.9) follows.

**Corollary 3.4.** Under the conditions of Lemma 3.3 (ii),

\[
(3.10) \quad \|Q_j f\|_p \leq c (1 + j)^{-\alpha\lambda}\|f\|_p, \quad j \geq 0, \quad f \in L^p,
\]

where \( \lambda/2 < 1/p < 1 - \lambda/2 \) if \( 0 \leq \lambda < 1, \quad p = 2 \) if \( \lambda = 1 \), and \( c \) is independent of \( j \).

**Proof:** Since \( Q_j \) is not a linear operator we cannot interpolate between (3.6) and (3.7) directly. Therefore we proceed as in [19], p. 280–281 (see also [8, p. 60]). Redenote \( f_{j,k} = Q_{j,k} f \) so that \( Q_j f = \sup_k |Q_{j,k} f| \) (cf. (3.5)). Let \( \mathcal{K} \) be the set of all measurable integer-valued functions \( k(x) \) on \( \mathbb{R}^n \). Given \( k(x) \in \mathcal{K} \), define a linear operator

\[
Q_{j,k(x)} f(x) = \mathcal{L}_\nu \int_{2^{j+k(x)}(x)}^2 dr \int_{\Sigma_{n-1}} f(x - r\theta) d\nu(\theta) - \int_{\mathbb{R}^n} f(x - y) dy \int_{2^{j+k(x)}(x)}^2 dr \int_{\Sigma_{n-1}} 2^{-nk(x)} \Phi(2^{-k(x)}(y - r\theta)) d\nu(\theta),
\]

so that

\[
(3.11) \quad \sup_{k \in \mathcal{K}} |Q_{j,k(x)} f(x)| = Q_j f(x).
\]
By (3.6), for \( f \in L^q \) we have

\[
\| \sup_{k \in \mathcal{K}} |Q_{j,k(x)}f| \|_q = \|Q_jf\|_q \leq c \|f\|_q \quad \forall q \in (1, \infty).
\]

Moreover,

\[
\sup_{k \in \mathcal{K}} \|Q_{j,k(x)}f\|_q = \|Q_jf\|_q.
\]

Indeed, by (3.11) there is a sequence \( \{k_l(x)\} \subset \mathcal{K} \) such that \( \lim_{l \to \infty} |Q_{j,k_l(x)}f(x)| = Q_jf(x) \), and therefore \( \lim_{l \to \infty} \|Q_{j,k_l(x)}f\|_q = \|Q_jf\|_q \) (use the Lebesgue theorem on dominated convergence together with (3.12)). The last equality implies (3.13) because \( \|Q_{j,k(x)}f\|_q \leq \|Q_jf\|_q \quad \forall k \in \mathcal{K} \).

Since (3.6) and (3.7) are valid for \( Q_{j,k(x)} \), then

\[
\|Q_{j,k(x)}f\|_p \leq c (1 + j)^{-\alpha \lambda} \|f\|_p,
\]

where \( c \) is independent of \( j \) and \( k(x) \), \( \lambda \) and \( p \) are as required. The relations (3.14) and (3.13) imply (3.10).

4. Proof of Theorem B.

**Step 1.** Let us prove (1.8) for \( f \in C_c^\infty(\mathbb{R}^n) \). Suppose that \( 2^{j-1} < \varepsilon < 2^j \), \( 2^{\ell-1} < \rho < 2^\ell \) for some \( j, \ell \in \mathbb{Z} \). Then

\[
T_{\nu}^{\varepsilon, \rho} f = T_{\nu}^{\infty, \infty} f - T_{\nu}^{\rho, \infty} f,
\]

\[
T_{\nu}^{\infty, \infty} f(x) = \sum_{k=j}^{\infty} (\omega_k * f)(x) + \int_{\varepsilon}^{2^j} \frac{dr}{r} \int_{\sum_{n=1}^{\infty}} f(x - r\theta) d\nu(\theta),
\]

and

\[
\sup_{0 < \varepsilon < \rho < \infty} |T_{\nu}^{\varepsilon, \rho} f(x)| \leq 2 \sup_{j \in \mathbb{Z}} \left| \sum_{k=j}^{\infty} (\omega_k * f)(x) \right| + 2 \sup_{j \in \mathbb{Z}} \left| (\omega_j * |f|)(x) \right|.
\]

By Lemma 2.3 (with \( s = p \)),

\[
\| \sup_{j \in \mathbb{Z}} (|\omega_j| * |f|) \|_p \leq c \|f\|_p \quad \forall p \in (1, \infty).
\]

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Let us estimate the first term in the right-hand side of (4.2). We take \( \Phi_j(x) = 2^{-jn} \Phi(2^{-j} x) \) with \( \Phi \) as in Lemma 3.3 (ii). Then

\[
(4.4) \quad \sup_{j \in \mathbb{Z}} \left| \sum_{k=j}^{\infty} \omega_k * f \right| \leq \sup_{j \in \mathbb{Z}} \left| (\delta - \Phi_j) * \left( \sum_{k=j}^{\infty} \omega_k * f \right) \right| + \sup_{j \in \mathbb{Z}} \left| \Phi_j * \left( \sum_{k=j}^{\infty} \omega_k * f \right) \right|,
\]

\( \delta \) being the Dirac delta function. By (3.5) and (3.10),

\[
(4.5) \quad \left\| \sup_{j \in \mathbb{Z}} \left( \delta - \Phi_j \right) * \left( \sum_{k=j}^{\infty} \omega_k * f \right) \right\|_p \leq \left\| \sum_{j=0}^{\infty} Q_j f \right\|_p \leq C \|f\|_p,
\]

provided (1.9). Furthermore, by Theorem A and Lemma 3.1 (choose \( \sigma_k = \omega_k \)),

\[
(4.6) \quad \left\| \sup_{j \in \mathbb{Z}} \left| \Phi_j * \left( \sum_{k=j}^{\infty} \omega_k * f \right) \right| \right\|_p \leq C \|f\|_p.
\]

These estimates imply (1.8).

**Step 2.** Suppose that \( f \in L^p \), \( \tilde{T}_\nu : L^p \to L^p \) is an extension of the operator \( T_\nu \), the existence of which was stated in Theorem A. Let us prove that \( \Delta = \|T_\nu^\varepsilon f - \tilde{T}_\nu f\|_p \to 0 \) as \( \varepsilon \to 0 \), \( \rho \to \infty \) for \( p \) satisfying (1.9). This result is a consequence of the uniform estimate

\[
(4.7) \quad \sup_{0 < \varepsilon < \rho < \infty} \|T_\nu^\varepsilon f\|_p \leq A \|f\|_p, \quad A = \text{const}.
\]

Indeed, if \( \{f_m\} \subset C_0^\infty \), \( \lim_{m \to \infty} \|f - f_m\|_p = 0 \), then

\[
\Delta \leq \|T_\nu^\varepsilon f - f_m\|_p + \|T_\nu^\varepsilon f_m - \tilde{T}_\nu f_m\|_p + \|\tilde{T}_\nu (f_m - f)\|_p \leq A \|f - f_m\|_p + \|T_\nu^\varepsilon f_m - \tilde{T}_\nu f_m\|_p + c \|f_m - f\|_p.
\]

The first and the last terms become small due to \( m \), the second term tends to 0 as \( \varepsilon \to 0 \) and \( \rho \to \infty \) by the Lebesgue theorem of dominated convergence which is applicable owing to Step 1.

In order to prove (4.7) we note that the uniform inequality \( \|T_\nu^\varepsilon \omega\|_p \leq A \|\omega\|_p \) holds for \( \omega \in C_0^\infty \) due to Step 1. Hence it can be extended to all \( f \in L^p \), and we get \( \|T_\nu^\varepsilon f\|_p \leq A \|f\|_p \). This gives (4.7).

**Step 3.** Let us prove (1.8) for \( f \in L^p \). It suffices to check (4.1)-(4.6) for such an \( f \). By the reasons, which are similar to [1], p. 292, the integral \( T_\nu^\varepsilon \omega \) is well-defined for a.e.
\[ x \in \mathbb{R}^n, \text{ which implies the a.e. convergence of the series } \sum_{k=j}^{\infty} \omega_k \ast f(x) \text{ for each } j \in \mathbb{Z}. \text{ This gives (4.1)-(4.4). The validity of (4.5) follows from Corollary 3.4. Let us check (4.6). Note that the relation } \Delta \to 0 \text{ in Step 2 implies } \lim_{m \to \infty} \sum_{k=t}^{m} \omega_k \ast f. \text{ Suppose for a moment that the series } A_j f \text{ and } B_j f = \sum_{k=0}^{j-1} \omega_k \ast f \text{ converge in the } L^p\text{-norm. Then } A_j f \overset{a.e.}{=} \tilde{T}_v f - B_j f, \text{ and the left-hand side of (4.6) does not exceed}

\begin{equation}
\| \sup_{j \in \mathbb{Z}} |\Phi_j \ast \tilde{T}_v f| \|_p + \| \sup_{j \in \mathbb{Z}} |\Phi_j \ast B_j f| \|_p, \tag{4.8}
\end{equation}

(the series } A_j f(x) \text{ being also convergent in the a.e. sense). By [17, p. 27] and Step 2, we can estimate the first term in (4.8):

\begin{equation}
\| \sup_{j \in \mathbb{Z}} |\Phi_j \ast \tilde{T}_v f| \|_p \leq c \| M(\tilde{T}_v f) \|_p \leq c \| \tilde{T}_v f \|_p \leq c \| f \|_p, \tag{4.9}
\end{equation}

where } M \text{ is the Hardy-Littlewood maximal operator.}

Let us estimate the second term in (4.8). By the reasons, which are similar to [19], p. 162-163, the series } B_j \Phi_j(x) \text{ converges for each } x. \text{ Furthermore, by (3.2) (with } \sigma_k = \omega_k, \text{ } |B_j \Phi_j(x)| \leq c \psi_j(x), \text{ } \psi_j(x) = 2^{-jn} / (1 + |2^{-j} x|)^{n+1}. \text{ Hence } \Phi_j \ast B_j f \overset{a.e.}{=} B_j \Phi_j \ast f \text{ (both functions belong to } L^p \text{ and coincide in the weak sense), and we obtain}

\begin{equation}
\| \sup_{j \in \mathbb{Z}} |\Phi_j \ast B_j f| \|_p \leq c \| \sup_{j \in \mathbb{Z}} (|\psi_j| \ast |f|) \|_p \leq c \| M f \|_p \leq c \| f \|_p. \tag{4.10}
\end{equation}

By (4.8)-(4.10) we get (4.6).

It remains to check the } L^p\text{-convergence of the series } A_j f \text{ and } B_j f. \text{ The operators } A_j \text{ and } B_j \text{ extend as } L^p\text{-bounded operators with the norms, independent of } j. \text{ To see this, one should use Lemmas 2.2 and 2.3 by putting } \sigma_k = \begin{cases} \omega_k & \text{if } k \geq j \\ 0 & \text{if } k < j \end{cases} \text{ for } A_j, \text{ and } \sigma_k = \begin{cases} 0 & \text{if } k \geq j \\ \omega_k & \text{if } k < j \end{cases} \text{ for } B_j. \text{ By Step 1, for } f \in C_c^\infty \text{ we have}

\begin{equation}
\| \sup_{m \in \mathbb{Z}} \left| \sum_{k=j}^{m} \omega_k \ast f \right| \|_p \leq \| \sup_{t,m} \left| \sum_{k=t}^{m} \omega_k \ast f \right| \|_p \leq \| \sup_{0<\varepsilon<\rho<\infty} |T_v^{\varepsilon,\rho} f| \|_p \leq c \| f \|_p.
\end{equation}

Hence, by the reasons, which are similar to those in Step 2, we obtain an } L^p\text{-convergence of } A_j f \text{ and } B_j f \text{ for } f \in L^p.

Thus, the maximal estimate (1.8) is proved. The a.e. convergence of } T_v^{\varepsilon,\rho} f \text{ then follows in a standard way (use, e.g., Theorem 3.12 from [19, Chapter II]).}
5. Singular integrals, generated by zonal spherical measures.

5.1 Auxiliary results.

**Lemma 5.1.** Suppose that \( n > 2 \), \( \Lambda \) is an \( SO(n-1) \)-invariant subset of \( \Sigma_{n-1} \), \( \nu \in M_2(\Sigma_{n-1}) \) (see Notation). If \( f \in L^1(\Sigma_{n-1}, d\nu) \), then

\[
\int_{\Sigma_{n-2}} f(\vartheta) d\nu(\vartheta) = \frac{1}{\sigma_{n-2}} \int_{\Lambda} d\vartheta \int_{\Sigma_{n-2}} f(\sqrt{1 - \vartheta_n^2 \sigma + \vartheta_n e_n}) d\sigma,
\]

where \( \sigma_{n-2} = |\Sigma_{n-1}| \) and \( d\sigma \) is the usual Lebesgue measure on \( \Sigma_{n-2} \).

**Proof.** Let \( \vartheta = (\sin \vartheta) \sigma + (\cos \vartheta) e_n, \sigma \in \Sigma_{n-2}, \cos \vartheta = \vartheta_n \). Then

\[
\int_{\Sigma_{n-2}} f(\vartheta) d\nu(\vartheta) = \int_{SO(n-1)} \int_{\Lambda} d\gamma \int_{\Sigma_{n-2}} f(\gamma \vartheta) d\nu(\vartheta) = \int_{\Lambda} d\vartheta \int_{SO(n-1)} f((\sin \vartheta) \gamma \sigma + (\cos \vartheta) e_n) d\gamma = \frac{1}{\sigma_{n-2}} \int_{\Lambda} d\vartheta \int_{\Sigma_{n-2}} f((\sin \vartheta) \sigma + (\cos \vartheta) e_n) d\sigma,
\]

which gives (5.1).

**Lemma 5.2.** Let \( \nu \) and \( \Lambda \) be the same as in Lemma 5.1. Then for \( \beta \in (0, 1/2) \) and \( n > 2 \),

\[
\sup_{|\xi|=1} \int_{\Lambda} |\theta \cdot \xi|^{-\beta} d|\nu|(\theta) \leq c \int_{\Lambda} |\theta_n|^{-\beta} (1 - \theta_n^2)^{-\beta/2} d|\nu|(\theta), \quad c = c(n, \beta).
\]

**Proof.** Let \( \xi = (\hat{\xi}, \xi_n) \), \( \hat{\xi} \in \mathbb{R}^{n-1} \). By (5.1),

\[
\int_{\Lambda} |\theta \cdot \xi|^{-\beta} d|\nu|(\theta) = \frac{1}{\sigma_{n-2}} \int_{\Lambda} d|\nu|(\theta) \int_{\Sigma_{n-2}} |1 - \theta_n^2 \sigma \cdot \hat{\xi} + \theta_n \xi_n|^{-\beta} d\sigma = \frac{\sigma_{n-3}}{\sigma_{n-2}} \int_{\Lambda} A(\xi, \theta) d|\nu|(\theta), \quad A(\xi, \theta) = \int_{-1}^1 \sqrt{1 - \theta_n^2} (1 - \xi_n^2 + \theta_n \xi_n) |1 - t^2|^{-\beta/2} dt.
\]

If \( |\theta_n| \geq 1 - \xi_n^2 \), then \( |\xi_n| \geq 1 - \theta_n^2 \), \( b \overset{\text{def}}{=} \sqrt{1 - \theta_n^2} (1 - \xi_n^2) / |\theta_n \xi_n| \leq 1 \), and we have

\[
A(\xi, \theta) \leq \int_{-1}^1 \frac{(1 - t^2)^{n-2}}{|\theta_n \xi_n| - |t| \sqrt{1 - \theta_n^2} (1 - \xi_n^2)} dt \leq |\theta_n \xi_n|^{-\beta} \int_{-1}^1 \frac{(1 - t^2)^{n-2}}{(1 - b |t|)^{\beta}} dt \leq 2 |\theta_n|^{-\beta} (1 - \theta_n^2)^{-\beta/2} \int_0^1 \frac{(1 - t^2)^{n-2}}{(1 - t)^\beta} dt = c |\theta_n|^{-\beta} (1 - \theta_n^2)^{-\beta/2}, \quad c = \text{const}.
\]
If $|\theta_n| < \sqrt{1 - \xi_n^2}$, i.e. $|\xi_n| < \sqrt{1 - \theta_n^2}$, then $a \overset{\text{def}}{=} -\theta_n \xi_n / \sqrt{(1 - \xi_n^2)(1 - \theta_n^2)} \in (-1, 1)$, and we get

$$A(\xi, \theta) \leq \int_{-1}^{1} \frac{(1 - t^2)^{n/2 - 2}}{|t - a|^\beta} \leq |\theta_n|^{-\beta} (1 - \theta_n^2)^{-\beta/2} [I(a) + I(-a)], \quad I(a) = \int_{-1}^{a} \frac{(1 - t^2)^{n/2 - 2}}{(a - t)^\beta} dt.$$  

(5.4)

By the formulas 2.2.6.1 from [10], and 9.102.2 from [5] we obtain

$$I(a) = \frac{2^{n/2 - 2} B(n/2 - 1, 1 - \beta)}{(a + 1)^{1 + \beta - n/2}} F \left( \frac{n}{2} - 1, 2 - \frac{n}{2}; a + 1; \frac{a + 1}{2} \right) \leq c(n, \beta) < \infty,$$

$0 < \beta < 1/2$. The same estimate holds for $I(-a)$.

**Lemma 5.3.** Let $\nu \in M_x(\Sigma_{n-1})$, $n > 2$. Then

$$\sup_{|\xi| = 1} \int_{\Sigma_{n-1}} \log \frac{1}{|\theta \cdot \xi|} d|\nu|(\theta) < \infty,$$

if and only if

$$\int_{\Sigma_{n-1}} \log \frac{1}{|\theta_n \sqrt{1 - \theta_n^2}|} d|\nu|(\theta) < \infty.$$  

PROOF. Denote

$$R(\xi_n, \theta_n) = \int_{-1}^{1} \frac{(1 - t^2)^{n/2 - 2}}{|t \sqrt{(1 - \theta_n^2)(1 - \xi_n^2) + \theta_n \xi_n}|} dt.$$  

As in (5.3) we have

$$\int_{\Sigma_{n-1}} \log \frac{1}{|\theta \cdot \xi|} d|\nu|(\theta) = \frac{\sigma_{n-3}}{\sigma_{n-2}} \int_{\Sigma_{n-1}} R(\xi_n, \theta_n)d|\nu|(\theta) \overset{\text{def}}{=} \frac{\sigma_{n-3}}{\sigma_{n-2}} K(\xi_n).$$

(5.7)

Using the same notation as in the proof of Lemma 5.2 we have

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for $|\theta_n| \geq \sqrt{1 - \xi_n^2}$:

$$
R(\xi_n, \theta_n) \leq \int_{-1}^{1} (1 - t^2)^{n/2-2} \log \frac{1}{|\theta_n \xi_n|(1 - |t|)} \, dt \leq \frac{c_1 \log \frac{1}{|\theta_n \xi_n|}}{(1 - |\theta_n \xi_n|)} + 2 \int_{0}^{1} (1 - t^2)^{n/2-2} \log \frac{1}{1 - t} \, dt \leq c_1 \log \frac{1}{|\theta_n \sqrt{1 - \theta_n^2}|} + c_2;
$$

for $|\theta_n| < \sqrt{1 - \xi_n^2}$:

$$
R(\xi_n, \theta_n) \leq \log \frac{1}{\sqrt{(1 - \xi_n^2)(1 - \xi_n^2)}} \left[ c_1 + \frac{1}{1} \int_{-1}^{1} (1 - t^2)^{n/2-2} \log \frac{1}{|t - a|} \, dt \right] < \log \frac{1}{|\theta_n \sqrt{1 - \theta_n^2}|} \left[ c_1 + c_2 \int_{1}^{1} (1 - t^2)^{n/2-2} \log \frac{1}{|t - a|^{1/4}} \, dt \right] \leq c \log \frac{1}{|\theta_n \sqrt{1 - \theta_n^2}|},
$$

c being independent of $a$ (see the estimate of the integral in (5.4)). Hence (5.6) implies (5.5). Conversely, if (5.5) holds, then (see (5.7)) $K(0) < \infty$ and $K(\pm 1) < \infty$. Since

$$
K(0) = \int_{\Sigma_{n-1}} \log \frac{1}{\sqrt{1 - \theta_n^2}} d|\nu|(|\theta|) \int_{-1}^{1} (1 - t^2)^{n/2-2} dt + \int_{\Sigma_{n-1}} d|\nu|(|\theta|) \int_{-1}^{1} (1 - t^2)^{n/2-2} \log \frac{1}{|t|} \, dt,
$$

and

$$
K(\pm 1) = \int_{\Sigma_{n-1}} \log \frac{1}{|\theta_n|^2} d|\nu|(|\theta|) \int_{-1}^{1} (1 - t^2)^{n/2-2} dt,
$$

then

$$
\int_{\Sigma_{n-1}} \log \frac{1}{\sqrt{1 - \theta_n^2}} d|\nu|(|\theta|) < \infty, \quad \int_{\Sigma_{n-1}} \log \frac{1}{|\theta_n|^2} d|\nu|(|\theta|) < \infty,
$$

and (5.6) follows.

The next result will be used in the proof of Theorem C.

**Theorem 5.4** (cf. [20], p. 3). Let $\{\sigma_j\}_{j \in \mathbb{Z}}$ be a sequence of finite Borel measures on $\mathbb{R}^n$, which for integers $m \geq 0$ admit a splitting $\sigma_j = U_j^m + L_j^m$ into Borel measures $U_j^m$ and $L_j^m$ so that

$$
U_j^m \quad \text{and} \quad L_j^m \quad \text{are supported in} \quad \{x : |x| < 2^j\};
$$

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\[(5.9) \quad \|L^m_j\| \leq c, \quad |\hat{L}^m_j(\xi)| \leq \frac{c2^{am}}{(2^j|\xi|)^\alpha}, \quad \alpha > 0;\]

\[(5.10) \quad \sup_j \sum_{m=0}^{\infty} \|U^m_j\| \leq c.\]

Here \(c\) and \(a\) are nonnegative constants, independent of \(m\) and \(j\). If the operator

\[(5.11) \quad Tf = \sum_{j=-\infty}^{\infty} \sigma_j * f, \quad f \in C_c^\infty(\mathbb{R}^n),\]

extends to a bounded operator on \(L^2(\mathbb{R}^n)\), then \(T\) extends to a bounded operator on \(L^p(\mathbb{R}^n), 1 < p < \infty.\)

5.2. Proof of Theorem C and Corollary 1.4.

We denote

\[(5.12) \quad \Gamma_m = \{\theta \in \Sigma_{n-1} : |\theta_n| \sqrt{1 - \theta_n^2} < 2^{-m}\}, \quad \Gamma^c_m = \Sigma_{n-1} \setminus \Gamma_m,\]

and set \(\sigma_j = U^m_j + L^m_j\), where the measures \(L^m_j\) and \(U^m_j\) are defined by

\[(5.13) \quad (L^m_j, g) = c_\nu \int_{\Gamma^c_m} \int_{2^j}^{2^{j+1}} \frac{dr}{r} g(r\theta) d\nu(\theta), \quad c_\nu = \frac{1}{\|\nu\| \log 2};\]

\[(5.14) \quad (U^m_j, g) = c_\nu \int_{\Gamma_m} \int_{2^j}^{2^{j+1}} \frac{dr}{r} g(r\theta) d\nu(\theta),\]

g \in C_0(\mathbb{R}^n). Suppose that \(f \in C_c^\infty(\mathbb{R}^n)\). Then the series \(T_\nu f(x) = \sum_{j \in \mathbb{Z}} (\sigma_j * f)(x)\), converges for each \(x \in \mathbb{R}^n\). By Lemma 5.3, and by the reasons, which are similar to [18, p. 40], \(T_\nu\) extends to a bounded operator on \(L^2(\mathbb{R}^n)\).

Thus, by Theorem 5.4, it suffices to check (5.8)–(5.10). The validity of (5.8) and the first condition in (5.9) is clear. To check the second inequality in (5.9), we note that

\[(L^m_j)^{\wedge}(\xi) = c_\nu \int_{\Gamma^c_m} d\nu(\theta) \left\{ \int_{2^j}^{2^{j+1}} e^{-2\pi i r \theta \xi} \frac{dr}{r} \right\}.\]
The integral in brackets is dominated by $\log 2$ and also by $2^{-j}|\theta \cdot \xi|^{-1}$. Hence, it does not exceed $c (2^{-j}|\theta \cdot \xi|)^{-\alpha}$ for any $\alpha \in (0, 1)$. Let $\alpha = 1/4$, $\xi' = \xi/|\xi|$. Then by Lemma 5.2,

$$|(L_j^m)^\ell(\xi)| \leq \frac{c}{(2^j|\xi|)^{1/4}} \int_{\Gamma_m} \frac{d|\nu|(|\theta|)}{|\theta \cdot \xi|^1} \leq \frac{c}{(2^j|\xi|)^{1/4}} \int_{\Gamma_m} \frac{d|\nu|(|\theta|)}{(|\theta_n|^2 - \theta_n^2)^{1/4}} \leq \frac{c}{(2^j|\xi|)^{1/4}}. $$

It remains to check (5.10). By (5.14),

$$\|U_j^m\| \leq c \left( \frac{2^{j+1}}{r} \int_{\Gamma_m} d|\nu|(|\theta|) \right) \leq c \int_{\Gamma_m} d|\nu|(|\theta|)$$

(see the proof of (2.11)). Hence (see (5.12))

$$\sum_{m=0}^{\infty} \|U_j^m\| \leq c \sum_{m=0}^{\infty} \int_{\Gamma_m} d|\nu|(|\theta|) = c \int_{\Sigma_{n-1}} d|\nu|(|\theta|) \left[ \sum_{m=0}^{\log_2(1/|\theta_n|/\sqrt{1-\theta_n^2})} 1 \right] \leq c \int_{\Sigma_{n-1}} \log \frac{1}{|\theta_n|/\sqrt{1-\theta_n^2}} d|\nu|(|\theta|) < \infty,$$

which gives (5.10). The statement (a) is proved. The statement (b) follows from Theorem B by taking into account that (1.6) holds for all $\alpha > 0$ owing to (1.11) and Lemma 5.2. A Corollary 1.4 is a consequence of part (a) of Theorem C and Lemma 5.3.

5.3. **Proof of Corollary 1.5.**

The required function can be constructed as follows. Denote

$$\Lambda_1 = \{\theta \in \Sigma_{n-1} : 1/4 < \theta_n < 3/4\}, \quad \Lambda_2 = \{\theta \in \Sigma_{n-1} : 1/3 < \theta_n \leq 1/2\} \subset \Lambda_1,$$

and let $\psi : \Sigma_{n-1} \to \mathbb{R}$ be an integrable even zonal function such that $\psi(\theta) > 0$ on $\Lambda_1$ and $|\psi| \log(1 + |\psi|) \notin L^1(\Lambda_2)$. We define

$$\Omega(\theta) = \begin{cases} 
\lambda |\theta_n|^{-1}(\log |\theta_n|^{-1})^{-2}(\log \log |\theta_n|^{-1})^{-2} & \text{if } |\theta_n| < 1/100, \\
\psi(\theta) & \text{if } 1/4 < |\theta_n| < 3/4, \\
0 & \text{otherwise,}
\end{cases}$$

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where $\lambda(< 0)$ is chosen so that $\int_{\Sigma_{n-1}} \Omega(\theta) d\theta = 0$. For all $\alpha > 0$,

$$\sup_{|\xi|=1} \int_{\Sigma_{n-1}} \left( \log \frac{1}{|\xi \cdot \theta|} \right)^{1+\alpha} |\Omega(\theta)| d\theta \geq \int_{\{\theta : |\theta|<1/10\}} \left( \log \frac{1}{|\theta_n|} \right)^{1+\alpha} |\Omega(\theta)| d\theta = \infty,$$

i.e. (1.4) fails. Let us prove that $\Omega \not\in H^1(\Sigma_{n-1})$. Assuming the contrary and setting $g(x) = |x|^{1-n}\Omega(x')$, $x' = x/|x|$, if $|x| \leq 2$, and $g(x) \equiv 0$ if $|x| > 2$, we obtain $g(x) \in H^1(\mathbb{R}^n)$ (see Lemma 2.5 from [15]). Since $g$ is positive on the open set

$$\tilde{\Lambda}_1 = \{ x = r\theta \in \mathbb{R}^n : 1/4 < r < 2, \theta \in \Lambda_1 \},$$

then $g$ belongs to the class $L \log L$ on any compact $K \subset \tilde{\Lambda}_1$ ([17], p. 128). By choosing $K = \{ x = r\theta \in \mathbb{R}^n : 1/2 \leq r \leq 1, \theta \in \Lambda_2 \}$ we get

$$\infty > \int_K g(x) \log(1 + g(x)) dx = \int_{1/2}^{1} \int_{\Lambda_2} \Omega(x') \log \left( 1 + \frac{\Omega(x')}{r^{n-1}} \right) dx' \geq \int_{\Lambda_2} \psi(x') \log(1 + \psi(x')) dx' = \infty$$

due to the choice of $\psi$. This contradiction shows that $\Omega \not\in H^1(\Sigma_{n-1})$.

It remains to note that for $d\nu(\theta) = \Omega(\theta) d\theta$ the operator $T_\nu$ extends to a bounded operator on $L^p(\mathbb{R}^n)$, $\forall p \in (1, \infty)$ according to Corollary 1.4 and Lemma 5.3.

5.4. Proof of Corollary 1.6.

Let $\Lambda_1, \Lambda_2$ and $\psi$ be the same as in the previous subsection. Consider the function

$$\Omega(\theta) = \begin{cases} 
\lambda & \text{if } 1/5 < |\theta_n| \leq 1/4, \\
\psi(\theta) & \text{if } 1/4 < |\theta_n| < 3/4, \\
0, & \text{otherwise},
\end{cases}$$

(5.15)

where $\lambda < 0$ is such that $\int_{\Sigma_{n-1}} \Omega(\theta) d\theta = 0$. By Lemma 5.2 the function (5.15) satisfies (1.4) for all $\alpha > 0$. On the other hand, $\Omega \not\in H^1(\Sigma_{n-1})$ (see the proof of Corollary 1.5). A

Proof of Proposition 1.7.

Let $x = (x_1, \bar{x}) \in \mathbb{R}^n$, $\bar{x} = (x_2, \ldots, x_n) \in \mathbb{R}^{n-1}$. We set $\Omega(x') = \int_0^{\infty} r^{n-1} g(rx')dr$, $x' = x/|x|$, where $g(x) = u(x_1)v(\bar{x})$,

$$u(x_1) = \begin{cases} 
x_1^{-1}(\log|x_1|^{-1})^{-1-\varepsilon} & \text{if } 0 < |x_1| < 1/100, \\
0 & \text{if } |x_1| > 1/100,
\end{cases}$$

$$0 < \varepsilon < 1.$$
\[ v(\tilde{x}) = \begin{cases} sgn\, x_2 & \text{if } 1/100 < |\tilde{x}| < 2/100, \\ 0 & \text{otherwise.} \end{cases} \]

It is clear that \[ \int_{\Sigma_{n-1}} \Omega(x')dx' = \int_{\mathbb{R}^n} g(x)dx = 0. \] Let us check that \( \Omega \in H^1(\Sigma_{n-1}) \). By Lemma 2.4 from [15], it suffices to show that \( g \in H^1(\mathbb{R}^n) \).

Since \( u \in H^1(\mathbb{R}) \) (see Section 6.2 of [17], p. 178), and \( v \in H^1(\mathbb{R}^{n-1}) \) (see Section 1.2 of [17], p. 92), there are Schwartz functions \( \Phi_1(x_1) \) and \( \Phi_2(\tilde{x}) \) with nonvanishing integrals such that

\[ (6.1) \quad \sup_{t>0} |(\Phi_1) * u(x_1)| \in L^1(\mathbb{R}), \quad \sup_{t>0} |(\Phi_2) * v(\tilde{x})| \in L^1(\mathbb{R}^{n-1}) \]

(cf. Theorem 1 from [17], p. 91). In view of (6.1), \( \sup_{t>0} |(\Phi_1 * g)(x)| \in L^1(\mathbb{R}^n) \), where \( \Phi(x) = \Phi_1(x_1)\Phi_2(\tilde{x}) \) is a Schwartz function. This gives \( g \in H^1(\mathbb{R}^n) \).

Let us check (1.12). We set \( a(\xi) = \int |\Omega(x')| \log(1/|\xi \cdot x'|) \, dx' \). Since \( g(x) = |g(x)| \, sgn\, x_1 \, sgn\, x_2 \), then \( |\Omega(x')| = \int_0^\infty r^{n-1} |g(rx')|dr \). Hence

\[ \sup_{|\xi|=1} a(\xi) \geq a(e_1) = \int_{\mathbb{R}^n} |g(x)| \log \frac{1}{|x_1|} \, dx = \int_{\mathbb{R}^n} |g(x)| \log \frac{1}{|x_1|} \, dx - \int_{\mathbb{R}^n} |g(x)| \log \frac{1}{|x|} \, dx \geq \infty \]

because the first integral in (6.2) is infinite and the second one does not exceed

\[ \int_{\mathbb{R}^{n-1}} |u(x_1)| \, dx_1 \int_{\mathbb{R}} |v(\tilde{x})| \log \frac{1}{|\tilde{x}|} \, d\tilde{x} < \infty. \]

Thus we are done.

**7. Examples.**

Below we give examples of singular non-zonal measures, which satisfy (1.6) for all \( \alpha > 0 \). For these measures all statements of Theorem B hold in the maximal range \( 1 < p < \infty \).

**Example 7.1** \((n = 2)\). Consider the distribution function \( C(x) \) of the middle third Cantor set on \([0,1]\) (see [14], p. 145). Let \( C_2(\pi) = 2\pi C(\pi/2\pi) \), so that \( C_2(\pi) = 0 \) and \( C_2(2\pi) = 2\pi \). By setting

\[ g(x) = \begin{cases} x, & x \in [0, 2\pi/6], \\ 2\pi/3 - x, & x \in [2\pi/6, 2\pi/3], \\ 0, & x \in [2\pi/3, 4\pi/3], \\ 4\pi/3 - x, & x \in [4\pi/3, 10\pi/6], \\ x - 2\pi, & x \in [10\pi/6, 2\pi], \end{cases} \]
we define an auxiliary measure $\sigma$ on $[0, 2\pi]$ by

$$\sigma(E) = \int_E d\psi(x), \quad \psi(x) = (g \circ C_{2\pi})(x),$$

$E$ being a Borel subset of $[0, 2\pi]$ (since $\psi$ is a function of bounded variation, this definition is correct). Let $h : [0, 2\pi] \to \Sigma_1$ be a canonical map so that $h(\theta) = (\cos \theta, \sin \theta) \in \Sigma_1$. We define the required measure $\nu$ on $\Sigma_1$ as an image of $\sigma$ under the mapping $h$. It is clear that $\nu(\Sigma_1) = \sigma([0, 2\pi]) = 0$. Moreover, one can readily check that the total variation $\frac{d}{0} V \psi$ of $\psi$ on $[0, x]$ coincides with $C(x)$. Hence for any interval $[a, b] \subset [0, 2\pi]$, we have

$$|\sigma|([a, b]) \overset{\text{def}}{=} \frac{b}{a} V \psi = V \psi - \frac{a}{b} V \psi = C(b) - C(a),$$

and therefore [14, p. 157]

$$(7.1) \quad |\sigma|([a, b]) \leq c (b - a)^{\log_3 2}.$$ 

Let us show that

$$(7.2) \quad \sup_{|\xi|=1} \int_{\Sigma_1} |\vartheta \cdot \xi|^{-\beta} d|\nu|(|\vartheta| < \infty$$

for all $\beta < \log_3 2$ (this implies (1.6) for all $\alpha > 0$). Fix $\xi = (\cos \varphi, \sin \varphi) \in \Sigma_1$, and $\varepsilon \in (0, 2^{-10})$. Suppose for a moment that

$$(7.3) \quad \int_{\Sigma_1} |\vartheta \cdot \xi|^{-\beta} d|\nu|(|\vartheta| \leq \int_0^{2\pi} |\cos(\varphi - \theta)|^{-\beta} d|\sigma|(|\theta|.)$$

The right-hand side of (7.3) is equal to

$$(7.4) \quad \left( \int_{A_1^1(\varphi)} + \int_{A_1^2(\varphi)} + \int_{A_1^3(\varphi)} \right) |\cos(\varphi - \theta)|^{-\beta} d|\sigma|(|\theta|) = I_1 + I_2 + I_3,$$

where

$$A_1^1(\varphi) = \{ \theta \in [0, 2\pi] : \pi/2 - \varepsilon < |\theta - \varphi| < \pi/2 + \varepsilon \},$$

$$A_1^2(\varphi) = \{ \theta \in [0, 2\pi] : 3\pi/2 - \varepsilon < |\theta - \varphi| < 3\pi/2 + \varepsilon \},$$

$$A_1^3(\varphi) = \{ \theta \in [0, 2\pi] : \pi/2 - \varepsilon < |\theta - \varphi| < \pi/2 + \varepsilon \},$$

$$A_1^4(\varphi) = \{ \theta \in [0, 2\pi] : 3\pi/2 - \varepsilon < |\theta - \varphi| < 3\pi/2 + \varepsilon \}.$$
\[ A^3_\varepsilon(\varphi) = [0, 2\pi] \setminus (A^1_\varepsilon(\varphi) \cup A^2_\varepsilon(\varphi)). \]

The third integral in (7.4) is dominated by \((\sin \varepsilon)^{-\beta} |\sigma|([0, 2\pi])\). Furthermore, by Theorem 1.15 from [9], p. 15,

\[
I_1 \leq c \int_{A^1_\varepsilon(\varphi)} |\theta - \varphi - \frac{\pi}{2}|^{-\beta} d|\sigma|(\theta) = \beta(\int_0^\varepsilon + \int_\varepsilon^\infty) |\sigma|([\theta - \varphi - \frac{\pi}{2} \leq t]) \frac{dt}{t^{\beta+1}}.
\]

Both integrals are dominated by a constant which is independent of \(\varphi\). For the second integral this is obvious. For the first one the statement holds due to estimate

\[
|\sigma|([\theta \in A^1_\varepsilon(\varphi) : |\theta - \varphi - \pi/2| \leq t]) \leq |\sigma|([\theta \in [0, 2\pi] : |\theta - \varphi - \pi/2| \leq t]) \leq c t^{\log_2 2}
\]

which follows from (7.1). For \(I_2\) the argument is similar.

It remains to prove (7.3). For \(N \in \mathbb{N}\) denote \(D_\xi(N) = \{\vartheta \in \Sigma_1 : |\vartheta \cdot \xi|^{-\beta} \leq N\}\), and let \(\{S^\xi_m(\vartheta)\}_{m=1}^\infty\) be a sequence of simple functions, such that for each \(\vartheta \in D_\xi(N)\),

\[ 0 \leq S^\xi_1 \leq \ldots \leq S^\xi_m \leq \ldots \leq |\vartheta \cdot \xi|^{-\beta} \text{ and } S^\xi_m(\vartheta) \to |\vartheta \cdot \xi|^{-\beta} \text{ as } m \to \infty. \]

By the reasons, which are similar to those in the proof of (2.11), we have \(|\nu|(E) \leq |\sigma|(h^{-1}(E)) \text{ } \forall E \in B(\Sigma_1)\). Hence,

\[
\int_{D_\xi(N)} |\vartheta \cdot \xi|^{-\beta} d|\nu|(\vartheta) \leq \lim_{m \to \infty} \int_{D_\xi(N)} S^\xi_m(\vartheta) d|\nu|(\vartheta) \leq \lim_{m \to \infty} \int_{h^{-1}(D_\xi(N))} (S^\xi_m \circ h)(\theta) d|\sigma|(\theta) = \int_{h^{-1}(D_\xi(N))} |\cos(\varphi - \theta)|^{-\beta} d|\sigma|(\theta),
\]

where \(h^{-1}(D_\xi(N)) = \{\theta \in [0, 2\pi] : |\cos(\varphi - \theta)|^{-\beta} \leq N\}\). Tending \(N\) to infinity, we obtain (7.3).

The next example is motivated by Corollary 4.3 from [3], p. 553.

**Example 7.2** \((n > 2)\). Define a measure \(\nu\) on \(\Sigma_{n-1}\) by

\[
\int_{\Sigma_{n-1}} g(\vartheta) d\nu(\vartheta) = \int_{\Gamma} g(y) \Omega(y) d\Gamma y, \quad \int_{\Gamma} \Omega(y) d\Gamma y = 0,
\]

where \(\Gamma = \{\vartheta \in \Sigma_{n-1} : \vartheta_n = 1/2\}\), \(\Omega \in L^q(\Gamma)\) for some \(q > 1\), \(g \in C(\Sigma_{n-1})\); \(d\Gamma y\) is the induced Lebesgue measure on \(\Gamma\). By the reasons, which are similar to those in the proof
of (7.3), and by Hölder’s inequality we have
\[
\int_{\Sigma_{n-1}} |\xi \cdot \vartheta|^{-\beta} d|\nu| (\vartheta) \leq \int_{\Gamma} |\xi \cdot y|^{-\beta} |\Omega(y)| d\gamma y \leq K^{1/p} \parallel \Omega \parallel_{L^q(\Gamma)},
\]
where \(1/p + 1/q = 1\) and \(K = \int_{\Gamma} |\xi \cdot y|^{-p\beta} d\gamma y\) is bounded uniformly in \(\xi\) for \(0 < \beta < 1/2\ p\) (cf. Lemma 5.2). By Theorem A the relevant singular integral operator \(T_\nu\) is bounded on \(L^p\) for all \(1 < p < \infty\).

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