Singular integrals, generated by spherical measures

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Abstract. In this paper we study the L^p-mapping properties of the Calderón-Zygmund type singular integral operator $T_\nu f(x) = \int_0^\infty dr/r \int_{\Sigma_{n-1}} f(x - r\theta)d\nu(\theta)$, depending on a finite Borel measure ν. In particular it is shown that the conditions $\nu(\Sigma_{n-1}) = 0$, $\sup_{|\xi|=1} \int_{\Sigma_{n-1}} \log (1/|\theta \cdot \xi|)d|\nu|(\theta) < \infty$ imply the L^p-boundedness of T_ν, $1 < p < \infty$ provided $n > 2$, and ν is zonal.

1. Introduction.

Let Σ_{n-1} denote the unit sphere in R^n, and let $\Omega \in L^1(\Sigma_{n-1})$, $\int_{\Sigma_{n-1}} \Omega(\theta)d\theta = 0$. Consider the Calderón-Zygmund singular integral operator

$$(T_\Omega f)(x) = \lim_{\substack{\varepsilon\to 0\\ \rho\to\infty}} (T_\Omega^{\varepsilon,\rho} f)(x) = \lim_{\substack{\varepsilon\to 0\\ \rho\to\infty}} \int_{\varepsilon<|y|<\rho} f(x-y)\frac{\Omega(y/|y|)}{|y|^n}dy, \tag{1.1}$$

arising in a variety of problems (we refer the reader to the books [18], [19], [8], [4], [17] and the survey article [6] for more background information).

It is well-known [1], that if $\Omega \in L^1(\Sigma_{n-1})$ is *odd*, then the limit in (1.1) exists in the L^p-norm and a.e., for all $f \in L^p(\mathrm{R}^n)$, $1 < p < \infty$. This is a consequence of the corresponding one-dimensional result and the method of rotations. The main difficulty is connected with the case of Ω *even*. The following result of W. Connett [2], F. Ricci and G. Weiss [11], is well-known (see also [21], [6], [15]).

Theorem 1.1. *Let $f \in L^p(\mathrm{R}^n), 1 < p < \infty$. If Ω belongs to the Hardy space $H^1(\Sigma_{n-1})$, and $\int_{\Sigma_{n-1}} \Omega(\theta)d\theta = 0$, then*

$$\| \sup_{0<\varepsilon<\rho<\infty} |T_\Omega^{\varepsilon,\rho} f| \, \|_p \le c_p \|f\|_p, \tag{1.2}$$

and the limit (1.1) exists in the L^p-norm and a.e.

We also mention the following L^2-result (cf. [18], p. 40).

Theorem 1.2. *If*

$$\int_{\Sigma_{n-1}} \Omega(\theta)d\theta = 0 \quad \text{and} \quad \sup_{|\xi|=1} \int_{\Sigma_{n-1}} |\Omega(\theta)| \, \log \frac{1}{|\theta\cdot\xi|} \, d\theta < \infty, \tag{1.3}$$

then T_Ω is bounded from $L^2(\mathrm{R}^n)$ into itself.

L. Grafakos and A. Stefanov [7] considered the class of functions $\Omega(\theta)$ satisfying the following conditions:

$$\int_{\Sigma_{n-1}} \Omega(\theta)d\theta = 0, \qquad \sup_{|\xi|=1} \int_{\Sigma_{n-1}} |\Omega(\theta)| \Big(\log \frac{1}{|\theta\cdot\xi|} \Big)^{1+\alpha} d\theta < \infty, \qquad \alpha > 0. \tag{1.4}$$

They showed that this class is different from $H^1(\Sigma_{n-1})$ and proved the following theorem.

Theorem 1.3 ([7]). *If* Ω *satisfies (1.4), then* T_Ω *extends to a bounded operator from* L^p *into itself for* $2-\alpha/(1+\alpha)<p<2+\alpha$. *If, moreover,* $\alpha>1$, *then (1.2) holds for* $2-(2+2\alpha)/(1+2\alpha)<p<2+(2\alpha-2)/3$.

The method of the proof of Theorem 1.3 is based on ideas, which were developed by J. Duoandikoetxea and J. L. Rubio de Francia [3]. The following questions were posed in [7, pp. 456, 457]:

Question 1. *Are the ranges of indices in Theorem 1.3 sharp?*

Question 2. *Does the conditions (1.3) imply the* L^p*- boundedness of* T_Ω *for some* $p\neq 2$ *?*

In this paper we extend the aforementioned ranges of indices and show that (1.3) implies the L^p-boundedness of T_Ω for all $p\in(1,\infty)$ in the case $n>2$ provided that Ω is zonal (i.e. invariant under all rotations about the x_n-axis). We also consider a generalization of T_Ω with Ω replaced by a finite Borel measure on Σ_{n-1}. More precisely, let $M(\Sigma_{n-1})$ be a space of all such measures. Given $\nu\in M(\Sigma_{n-1})$, consider the singular integral operator

$$(T_\nu f)(x)=\lim_{\substack{\varepsilon\to 0\\ \rho\to\infty}}(T_\nu^{\varepsilon,\rho}f)(x)=\lim_{\substack{\varepsilon\to 0\\ \rho\to\infty}}\int_\varepsilon^\rho\frac{dr}{r}\int_{\Sigma_{n-1}}f(x-r\theta)d\nu(\theta). \tag{1.5}$$

If ν is absolutely continuous with respect to the Lebesgue measure $d\theta$ on Σ_{n-1} , i.e. $d\nu(\theta)=\Omega(\theta)d\theta$, $\Omega\in L^1(\Sigma_{n-1})$, then (1.5) coincides with (1.1).

Let us state our main results. The following theorems are related to Question 1.

Theorem A. *Let*

$$\nu(\Sigma_{n-1})=0,\qquad \sup_{|\xi|=1}\int_{\Sigma_{n-1}}\Big(\log\frac{1}{|\theta\cdot\xi|}\Big)^{1+\alpha}d|\nu|(\theta)<\infty\quad \textit{for some}\quad \alpha>0. \tag{1.6}$$

Then the operator T_ν , *initially defined by (1.5) on functions* $f\in C_c^\infty(\mathrm{R}^n)$, *extends to a linear bounded operator from* L^p *into itself provided*

$$\Big|\frac{1}{2}-\frac{1}{p}\Big|<\frac{\alpha}{2(1+\alpha)}. \tag{1.7}$$

Theorem B. *Suppose that $f \in L^p(\mathrm{R}^n)$, and $\nu \in M(\Sigma_{n-1})$ satisfies (1.6) for some $\alpha > 1$. Then*

$$\| \sup_{0<\varepsilon<\rho<\infty} |T_\nu^{\varepsilon,\rho} f| \, \|_p \le c_p \, \|f\|_p, \tag{1.8}$$

provided

$$\left| \frac{1}{2} - \frac{1}{p} \right| < \frac{\alpha - 1}{2\alpha}, \tag{1.9}$$

and the limit in (1.5) exists in the L^p-norm, and in the a.e. sense.

As in [7] and in many other papers, related to singular integral operators, we imploy the ideas developed by J. Duoandikoetxea and J. L. Rubio de Francia in [3]. The possibility of extending the bounds for p is based on the use of the method of rotations, instead of a "bootstrap" argument (cf. [17], p. 463), which was used in [7], p. 460.

Our next result concerns Question 2. Let $M_z(\Sigma_{n-1})$ be the subspace of $M(\Sigma_{n-1})$, consisting of *zonal* measures.

Theorem C. *Suppose that $\nu \in M_z(\Sigma_{n-1})$, $\nu(\Sigma_{n-1}) = 0$, $n > 2$.*

(a) *If*

$$\int_{\Sigma_{n-1}} \log \frac{1}{|\theta_n| \sqrt{1-\theta_n^2}} d|\nu|(\theta) < \infty, \tag{1.10}$$

then T_ν extends to a bounded operator from L^p into itself for all $p \in (1, \infty)$.

(b) *Let $f \in L^p(\mathrm{R}^n), 1 < p < \infty$. If*

$$\int_{\Sigma_{n-1}} |\theta_n|^{-\beta} (1 - \theta_n^2)^{-\beta/2} d|\nu|(\theta) < \infty \quad for\ some\ \beta \in (0, 1/2), \tag{1.11}$$

then (1.8) holds, and the limit in (1.5) exists in the L^p-norm and in the a.e. sense.

The proof of part (a) of this theorem employs recent results of D. K. Watson [20].

Corollary 1.4 (cf. Theorem 1.2). *Let $n > 2$, and let $\nu \in M_z(\Sigma_{n-1})$ satisfy (1.6) with $\alpha = 0$. Then T_ν extends to a bounded operator from L^p into itself for all $p \in (1, \infty)$.*

Corollary 1.5. *Let $n > 2$. There is an even function $\Omega \notin H^1(\Sigma_{n-1})$ which satisfies (1.3) and does not satisfy (1.4) for any $\alpha > 0$, but, nevertheless, the relevant operator T_Ω extends to bounded operators from L^p into itself for all $p \in (1, \infty)$.*

The above corollary shows that the ranges of indices in (1.7) are also not sharp.

Corollary 1.6. *Let $n > 2$. There is an even function $\Omega \notin H^1(\Sigma_{n-1})$, which satisfies (1.4) for all $\alpha > 0$.*

This result was proved in [7] for $n = 2$. But the proof, given there, was fairly complicated. We show that (for $n > 2$) examples of functions indicated in Corollary 1.6, can be easily obtained from Theorem C and geometric properties of the Hardy spaces $H^1(\Sigma_{n-1})$ and $H^1(\mathrm{R}^n)$.

We do not know if the results of Theorem C and Corollaries 1.4, 1.5 are true in the case $n = 2$. Another open problem is whether Corollary 1.4 holds for non-zonal ν if $p \neq 2$.

The following observation related to Theorems 1.1 and 1.2 is also of interest. Namely, the second condition in (1.3) may fail, but nevertheless, T_Ω is bounded on L^p for all $1 < p < \infty$. More precisely, the following statement holds.

Proposition 1.7. *There is an even function $\Omega \in H^1(\Sigma_{n-1})$ such that $\int\limits_{\Sigma_{n-1}} \Omega(\theta)d\theta = 0$ and*

$$\sup_{|\xi|=1} \int_{\Sigma_{n-1}} |\Omega(\theta)| \log \frac{1}{|\xi \cdot \theta|} \, d\theta = \infty. \tag{1.12}$$

The paper is organized as follows. In section 2 we prove Theorem A. Sections 3 and 4 are devoted to the proof of Theorem B. The proof of Theorem C and Corollaries 1.4-1.6 is given in section 5. In section 6 we prove Proposition 1.7 and in section 7 give examples of *non-zonal singular* measures, satisfying (1.6) for all $\alpha > 0$.

Notation. Let $\Sigma_{n-1} = \{x \in \mathrm{R}^n : |x| = 1\}$, $\sigma_{n-1} = |\Sigma_{n-1}| = 2\pi^{n/2}/\Gamma(n/2)$; $M(X)$ denotes the space of all finite Borel measures on a measure space X; $|\mu|$ designates the total variation of $\mu \in M(X)$. The notation $L^p(X)$ is standard; $C_0(\mathrm{R}^n)$ denotes the space of continuous on R^n functions, tending to zero at infinity; $C_c^\infty(\mathrm{R}^n)$ is the space of infinitely differentiable on R^n functions, having a compact support. We define the Fourier transform of $\mu \in M(\mathrm{R}^n)$ by $\hat{\mu}(\xi) = \int_{\mathrm{R}^n} e^{-2\pi i x\cdot\xi} d\mu(x)$. The group of rotations leaving the x_n-axis fixed will be denoted by $SO(n-1)$; $e_n = (0, \ldots, 0, 1)$. A measure $\nu \in M(\Sigma_{n-1})$ is called *zonal* if $\int_{\Sigma_{n-1}} f(\gamma\vartheta)d\nu(\vartheta) = \int_{\Sigma_{n-1}} f(\vartheta)d\nu(\vartheta)$ for each $\gamma \in SO(n-1)$ and each $f \in L^1(\Sigma_{n-1}, d\nu)$. The set of all zonal measures on Σ_{n-1} is denoted by $M_z(\Sigma_{n-1})$. The letter c designates a constant, not necessarily the same at each occurrence.

2. Proof of Theorem A.

We begin by proving some auxiliary statements. Following [3], let $\{\psi_j\}_{j\in \mathrm{Z}}$ be a smooth partition of the unity on $(0,\infty)$ so that

a) $\psi_j \in C^1(\mathrm{R}_+)$, $\quad 0\le \psi_j \le 1$, $\quad \sum_{j\in \mathrm{Z}} \psi_j^2(t) = 1$,

b) $\mathrm{supp}(\psi_0) \subseteq \{t\in \mathrm{R} : 1/2 \le t \le 2\}$, $\quad \psi_j(t) = \psi_0(2^j t)$,

c) $\psi_0(t) \equiv 1 \quad \forall t \in [1, 3/2]$, $\quad |\psi_j'(t)| \le c/t$.

Suppose also that $\sigma_k (\in M(\mathrm{R}^n))$, $k\in \mathrm{Z}$, is a sequence of measures such that

$$\|\sigma_k\| \le 1, \quad \mathrm{supp}\ \sigma_k \subseteq \{x \in \mathrm{R}^n : 2^k \le |x| \le 2^{k+1}\}, \tag{2.1}$$

and

$$|\hat{\sigma}_k(\xi)| \le \begin{cases} c\,|2^k\xi| & \text{if } |2^k\xi| \le 2, \\ c\, \log^{-1-\alpha} |2^k\xi| & \text{if } |2^k\xi| > 2, \quad \alpha > 0. \end{cases} \tag{2.2}$$

For $f \in C_c^\infty(\mathrm{R}^n)$, we define

$$Tf(x) = \sum_{k\in \mathrm{Z}} (\sigma_k * f)(x), \tag{2.3}$$

$$(S_j f)^\wedge(\xi) = \hat{f}(\xi)\psi_j(|\xi|), \quad (T_j f)(x) = \sum_{k\in \mathrm{Z}} S_{j+k}(\sigma_k * S_{j+k} f)(x), \quad j \in \mathrm{Z}. \tag{2.4}$$

Lemma 2.1. *Let* $f \in C_c^\infty(\mathrm{R}^n)$, *and*

$$\| \sup_{k\in \mathrm{Z}} (|\sigma_k| * |f|)\, \|_s \le c\, \|f\|_s \qquad \forall s \in (1,\infty). \tag{2.5}$$

Then

$$\|T_j f\|_q \le c\, \|f\|_q \quad \textit{for all } q \in (1,\infty). \tag{2.6}$$

If, moreover, σ_k, $k\in \mathrm{Z}$, *satisfy (2.2), then for all* $\lambda \in [0,1]$,

$$\|T_j f\|_p \le c\, (1+|j|)^{-(1+\alpha)\lambda} \|f\|_p, \tag{2.7}$$

provided that $\lambda/2 < 1/p < 1-\lambda/2$ *if* $0\le\lambda<1$, *and* $p=2$ *if* $\lambda = 1$. *The constant* c *in (2.6) and (2.7) is independent of* j.

The estimate (2.7) for the smaller range of p's was proved in [7], p. 460 (it was a consequence of a "bootstrap argument" and the assumption that (2.5) holds for $s=2$).

But the point is that in the studying of the operator (1.5), we can always assume that the maximal estimate, corresponding to (2.5) holds for a full range of s (see (2.13)).

PROOF OF LEMMA 2.1. The estimate (2.6) was established in [3], p. 545, and we recall its proof for convenience of the reader. We have

$$\|T_j f\|_q \overset{(1)}{\le} c\, \|(\sum_{k\in Z} |S_{j+k}(\sigma_k * S_{j+k} f)|^2)^{1/2}\|_q \overset{(2)}{\le} c\, \|(\sum_{k\in Z} |\sigma_k * S_{j+k} f|^2)^{1/2}\|_q \le$$

$$\overset{(3)}{\le} c\, \|(\sum_{k\in Z} |S_{j+k} f|^2)^{1/2}\|_q \overset{(4)}{\le} c\, \|f\|_q.$$

Here (1) and (4) follow from the Littlewood-Paley theory [17], p. 267; (2) is a special case of the more general estimate (4) from [13]; (3) holds according to the lemma on p. 544 from [3]. Furthermore, by Plancherel's theorem,

$$\|T_j f\|_2 \le \sum_{k\in Z} \|\hat\sigma_k \psi_{j+k}^2 \hat f\|_2 \le \sum_{k\in Z} \Big(\int\limits_{\text{supp } \psi_{j+k}} |\hat\sigma_k(\xi)|^2 |\hat f(\xi)|^2 d\xi \Big)^{1/2}.$$

Owing to (2.2), this gives (2.7) for $\lambda = 1,\ p = 2$ (cf. formula (11) from [7]). For $\lambda = 0$, (2.7) coincides with (2.6). The result for $0 < \lambda < 1$ follows by interpolation. Λ

Lemma 2.2. *Suppose that $\sigma_k \in M(\mathrm{R}^n), k \in \mathrm{Z}$, satisfy (2.1), (2.2), and (2.5). Then (2.3) extends to a linear bounded operator on $L^p(\mathrm{R}^n)$ provided $\Big|1/2 - 1/p\Big| < \alpha\ (2(1+\alpha))^{-1}$.*

PROOF. As in [3, p. 545], for $f \in C_c^\infty(\mathrm{R}^n)$ we have:

$$Tf = \sum_{k\in Z} \sigma_k * f = \sum_{j\in Z} T_j f, \tag{2.8}$$

and (2.7) yields $\|Tf\|_p \le c\, \|f\|_p \sum_j (1 + |j|)^{-(1+\alpha)\lambda}$, $\lambda/2 < 1/p < 1 - \lambda/2$. Assuming $\lambda > (1+\alpha)^{-1}$, we obtain the required result. Λ

Now we pass to the operator T_ν from (1.5). One can write $T_\nu f = \sum_{k\in Z} \omega_k * f$, where $\omega_k \in M(\mathrm{R}^n)$ are defined by

$$\int\limits_{\mathrm{R}^n} g(y) d\omega_k(y) = c_\nu \int\limits_{2^k}^{2^{k+1}} \frac{dr}{r} \int\limits_{\Sigma_{n-1}} g(r\theta) d\nu(\theta), \qquad c_\nu = (|\nu|(\Sigma_{n-1}) \log 2)^{-1}, \tag{2.9}$$

$g \in C_0(\mathrm{R}^n)$, and ν satisfies (1.6). Denote by $\mathcal{B}(\mathrm{R}^n)$ the σ-algebra of all Borel measurable sets in R^n. As usual [14, p. 116],

$$|\omega_k|(E) = \sup_{\{A_i\}} \sum_{i=1}^{\infty} |\omega_k(A_i)|, \qquad E \in \mathcal{B}(\mathrm{R}^n), \tag{2.10}$$

where the supremum is taken over all partitions of E by $A_i \in \mathcal{B}(\mathrm{R}^n)$.

Lemma 2.3. *Let $E \in \mathcal{B}(\mathrm{R}^n)$, $1_E(x) = 1$ for $x \in E$ and $1_E(x) = 0$ otherwise. Then*

$$|\omega_k|(E) \leq c_\nu \int_{2^k}^{2^{k+1}} \frac{dr}{r} \int\limits_{\Sigma_{n-1}} 1_E(r\theta) d|\nu|(\theta) \leq 1. \tag{2.11}$$

Furthermore, for $f \in L^s(\mathrm{R}^n)$, $1 < s < \infty$, the following relations hold:

$$(|\omega_k| * |f|)(x) \overset{a.e}{\leq} c_\nu \int\limits_{\Sigma_{n-1}} d|\nu|(\theta) \int\limits_{2^k}^{2^{k+1}} |f(x - r\theta)| \frac{dr}{r}, \tag{2.12}$$

$$\| \sup_{k \in \mathrm{Z}} (|\omega_k| * |f|) \|_s \leq c \, \|f\|_s. \tag{2.13}$$

Proof. The first relation follows by (2.9), (2.10):

$$|\omega_k|(E) \leq c_\nu \sup_{\{A_i\}} \sum_i \int_{2^k}^{2^{k+1}} \frac{dr}{r} \int\limits_{\Sigma_{n-1}} 1_{A_i}(r\theta) d|\nu|(\theta) \leq$$

$$\leq c_\nu \int_{2^k}^{2^{k+1}} \frac{dr}{r} \int_{\Sigma_{n-1}} \Big(\sup_{\{A_i\}} \sum_i 1_{A_i}(r\theta)\Big) d|\nu|(\theta) = c_\nu \int\limits_{2^k}^{2^{k+1}} \frac{dr}{r} \int\limits_{\Sigma_{n-1}} 1_E(r\theta) d|\nu|(\theta) \leq 1.$$

Furthermore, if $f \in C_0(\mathrm{R}^n)$, $f^x(y) = |f(x-y)|$, then by Theorem 1.17 from [14, p. 15] there is a sequence $\{S_m^x(y)\}_{m=1}^{\infty}$ of simple functions, such that $0 \leq S_1^x \leq S_2^x \leq \ldots \leq S_m^x \leq \ldots \leq |f^x|$ and $S_m^x(y) \to |f^x(y)|$ for each x and y. Hence

$$(|\omega_k| * |f|)(x) = (|\omega_k|, \lim_{m\to\infty} S_m^x) = \lim_{m\to\infty} (|\omega_k|, S_m^x) \overset{(2.11)}{\leq}$$

$$\leq \lim_{m\to\infty} c_\nu \int\limits_{2^k}^{2^{k+1}} \frac{dr}{r} \int\limits_{\Sigma_{n-1}} S_m^x(r\theta) d|\nu|(\theta) = c_\nu \int\limits_{2^k}^{2^{k+1}} \frac{dr}{r} \int\limits_{\Sigma_{n-1}} |f(x - r\theta)| d|\nu|(\theta).$$

In the general case $f \in L^s(\mathrm{R}^n)$, (2.12) then follows by the limiting argument from its validity for any convolution $(|f| * g_t)(x)$, $g_t(x) = t^{-n} g(x/t)$, $g \in C_c^\infty(\mathrm{R}^n)$, $g \geq 0$.

To prove (2.13), we denote by

$$(M_\theta f)(x) = \sup_{R>0} \frac{1}{R} \int_0^R |f(x - r\theta)| dr$$

the one-dimensional Hardy-Littlewood maximal operator in direction $\theta \in \Sigma_{n-1}$. By the method of rotations

$$\|M_\theta f\|_s \leq c\, \|f\|_s, \quad s > 1, \tag{2.14}$$

c being independent of θ. Then

$$(|\omega_k| * |f|)(x) \overset{(2.12)}{\leq} 2c_\nu \int_{\Sigma_{n-1}} \Big[2^{-k-1} \int_{2^k}^{2^{k+1}} |f(x - r\theta)| dr\Big] d|\nu|(\theta) \leq 2c_\nu \int_{\Sigma_{n-1}} (M_\theta f)(x) d|\nu|(\theta),$$

and the result follows by (2.14). Λ

Lemma 2.4. *Let $\nu \in M(\Sigma_{n-1})$ satisfy (1.6). Then there is a constant $c = c(\alpha, \nu) > 0$ such that for all $k \in \mathrm{Z}$,*

$$|\hat{\omega}_k(\xi)| \quad \leq \quad \begin{cases} c\, |2^k \xi| & \textit{if } \; |2^k \xi| \leq 2, \\ c\, \log^{-1-\alpha} |2^k \xi| & \textit{if } \; |2^k \xi| > 2. \end{cases} \tag{2.15}$$

This statement resembles the estimate (10) from [7]; see also [3, p. 550]. For convenience of the reader we prove (2.15) by completing some details, which were omitted in [7].

Proof. Since $\hat{\omega}_k(\xi) = \hat{\omega}_0(2^k \xi)$, it suffices to consider $k = 0$. The inequality

$$|\hat{\omega}_0(\xi)| = \Big| c_\nu \int_1^2 \frac{dr}{r} \int_{\Sigma_{n-1}} e^{-2\pi i r\theta\cdot\xi} d\nu(\theta) \Big| \leq c\, |\xi|, \quad |\xi| \leq 2,$$

is clear because $\nu(\Sigma_{n-1}) = 0$. The inequality $|\hat{\omega}_0(\xi)| \leq c\, (\log|\xi|)^{-1-\alpha}$, $|\xi| > 2$, follows by (1.6) from the estimate

$$A \overset{\text{def}}{=} \Big| \int_1^2 e^{-2\pi i r\theta\cdot\xi} \frac{dr}{r} \Big| \leq c \Big(\frac{b}{a}\Big)^\gamma, \quad a = \log|\xi|, \quad b = \log \frac{3/2}{|\theta \cdot \xi'|}, \tag{2.16}$$

$c = c(\gamma) = \text{const}$. The latter holds for all $\gamma \geq 0$ (in our case $\gamma = 1+\alpha$), $\theta\cdot\xi \neq 0$, $\xi' = \xi/|\xi|$.

Let us prove (2.16). Integration by parts yields

$$A = \Big|\int_1^2 \frac{d(e^{-2\pi i r\theta\cdot\xi})}{2\pi i r\theta\cdot\xi}\Big| \leq \frac{1}{\pi|\theta\cdot\xi|} \leq \frac{3/2}{|\theta\cdot\xi|} = e^{b-a}.$$

Note also that $b > \log(3/2) > 1/4$, i.e. $1 < 4b$. If $a-b \geq 1$, then $a/(a-b) \leq 1+b \leq 5b$, and therefore

$$A \leq \frac{(a-b)^\gamma e^{-(a-b)}}{(a-b)^\gamma} \leq \frac{c_\gamma}{(a-b)^\gamma} \leq c_\gamma \left(\frac{5b}{a}\right)^\gamma.$$

If $a-b<1$, then $a/b \leq (b+1)/b = 1+1/b < 5$ and we get $A \leq \int_1^2 dr/r < 1 < (5b/a)^\gamma$. Λ

It remains to note that Theorem A is a consequence of Lemmas 2.2 - 2.4 (put $\sigma_k = \omega_k$, where ω_k are defined by (2.9)).

3. Auxiliary statements.

Suppose that T is the operator (2.3), $\Phi(x)$ is a Schwartz function and $\Phi_j(x) = 2^{-jn}\Phi(2^{-j}x)$.

Lemma 3.1. *Let $f \in C_c^\infty(\mathrm{R}^n)$. If $\|Tf\|_p \leq c\,\|f\|_p$, then*

$$\| \sup_{j\in \mathrm{Z}} |\Phi_j * \sum_{k=j}^{\infty} \sigma_k * f|\,\|_p \leq c\,\|f\|_p. \tag{3.1}$$

The proof of this statement is given in [3], p. 548, and employs the estimate

$$\Big|\sum_{k=-\infty}^{j-1} (\sigma_k * \Phi_j)(y)\Big| = \Big|\Big(\Phi_j * \sum_{k=-\infty}^{j-1} \sigma_k\Big)(y)\Big| \leq c\,\psi_j(y), \tag{3.2}$$

$\psi_j(y) = 2^{-jn}/(1+|2^{-j}y|)^{n+1}$. Since this estimate will be used below and the proof of it was skipped in [3], we complete the details. For $x \in \mathit{supp}\ \sigma_k$, $y \in \mathrm{R}^n$, define $h_{x,y}(t) = \Phi_j(y-tx)$, $t \in [0,1]$. By (2.2), $\hat\sigma_k(0) = \sigma_k(\mathrm{R}^n) = 0$. Hence the left hand side of (3.2) does not exceed

$$\sum_{k=-\infty}^{j-1} \int_{\mathrm{R}^n} |h'_{x,y}(\eta)|d|\sigma_k|(x) \leq \sum_{k=-\infty}^{j-1} \int_{\mathrm{R}^n} \sum_{l=1}^{n} \Big|\frac{\partial\Phi_j}{\partial\xi_l}(y-\eta x)\Big||x_l|d|\sigma_k|(x),$$

where $\eta = \eta(x,y) \in [0,1]$, $\xi_l = y_l - \eta_l x$. The above expression is estimated by

$$c \sum_{k=-\infty}^{j-1} 2^{-j(n+1)} \int\limits_{2^k<|x|<2^{k+1}} (1+|(y-\eta x)2^{-j}|)^{-n-1}|x|d|\sigma_k|(x) \le$$

$$\le c \sum_{k=-\infty}^{j-1} 2^{-j(n+1)} \int\limits_{2^k<|x|<2^{k+1}} \left(\frac{1+|\eta x 2^{-j}|}{1+|y2^{-j}|}\right)^{n+1} |x|d|\sigma_k|(x) \le c\, 2^{n+1} \sum_{k=-\infty}^{j-1} 2^{-j+k+1}\psi_j(y),$$

which gives (3.2).

We recall Cotlar's lemma, which will be used below.

Lemma 3.2 ([17], p. 280). *Suppose that $\{Q_\ell\}$ is a finite collection of bounded operators on $L^2(\mathrm{R}^n)$. Assume that we are given a sequence of positive constants $\{\gamma(\ell)\}_{\ell\in \mathrm{Z}}$ with*

$$A = \sum_{\ell\in\mathrm{Z}} \gamma(\ell) < \infty, \tag{3.3}$$

and

$$\|Q_i^* Q_k\| \le [\gamma(i-k)]^2, \quad \|Q_i Q_k^*\| \le [\gamma(i-k)]^2; \tag{3.4}$$

here $\|\cdot\|$ denotes the operator norm on L^2. Then the operator $Q = \sum_\ell Q_\ell$ satisfies $\|Q\| \le A$.

For $\Phi_k(x)$ as in Lemma 3.1 and for ω_k defined by (2.9), we get

$$Q_j f = \sup_{k\in\mathrm{Z}} |f_{j,k}|, \qquad f_{j,k} = \omega_{j+k} * f - \Phi_k * \omega_{j+k} * f. \tag{3.5}$$

Lemma 3.3. *Let $j \ge 0$. There is a constant $c > 0$, independent of j, with the following properties.*

(i) *If $f \in L^q(\mathrm{R}^n)$, $1 < q < \infty$, then*

$$\|Q_j f\|_q \le c\, \|f\|_q. \tag{3.6}$$

(ii) *If Φ is a radial function, such that $|\hat{\Phi}(\xi)| \le 1$, $\hat{\Phi}(\xi) = 1$ for $|\xi| \le 2$ and $\hat{\Phi}(\xi) = 0$ for $|\xi| > 3$, and $f \in L^2(\mathrm{R}^n)$, then*

$$\|Q_j f\|_2 \le c\,(1+j)^{-\alpha}\|f\|_2, \tag{3.7}$$

provided that ν *satisfies (1.6) with* $\alpha > 1$.

PROOF. (i) We have $Q_j f \le \sup_{k\in Z}|\omega_{j+k} * f| + c\, M[\sup_{k\in Z}|\omega_{j+k} * f|]$, where M is the Hardy-Littlewood maximal operator. Hence (3.6) follows by Lemma 2.3.

(ii) Since $(\sup_k|f_{j,k}|)^2 \le \sup_k|f_{j,k}|^2 \le \sum_k |f_{j,k}|^2$, then

$$\|Q_j f\|_2^2 \le \|(\sum_{k\in Z}|f_{j,k}|^2)^{1/2}\|_2^2 = \lim_{N\to\infty}\sum_{|k|\le N}\int_{R^n}|f_{j,k}(x)|^2dx.$$

It suffices to show that for an arbitrary $N \in N$,

$$\sum_{|k|\le N}\int_{R^n}|f_{j,k}(x)|^2dx \le c\,(1+j)^{-2\alpha}\|f\|_2^2, \tag{3.8}$$

where c is independent of j and N. To prove (3.8) we make use of Lemma 3.2. Let $\{r_\ell(t)\}_{\ell=1}^\infty$ be an orthonormal system of the Rademacher functions in $L^2[0,1]$ so that $r_\ell(t) = sgn \sin 2^\ell t\pi$,

$$\sum_{|k|\le N}\int_{R^n}|f_{j,k}(x)|^2dx = \int_0^1 dt\int_{R^n}\Big|\sum_{|k|\le N} r_{k+N+1}(t)f_{j,k}(x)\Big|^2dx,$$

(cf. [22], p. 176, 180). Fix $N\in N$, $t\in[0,1]$, and set $Q^j_{k,N}f = r_{k+N+1}(t)f_{j,k}$. We claim that,

$$\|(Q^j_{i,N})^*Q^j_{k,N}\|_{2\to2} \le [\gamma_j(k-i)]^2, \quad \gamma_j(k-i) = c\,\frac{(1+j)^{-(1+\alpha)/2}}{(1+j+|k-i|)^{(1+\alpha)/2}}, \tag{3.9}$$

where c is independent of r_{k+N+1} and N (the same estimate holds for $Q^j_{i,N}(Q^j_{k,N})^*$).

Suppose for a moment, that (3.9) is true. Then

$$\sum_{\ell\in Z}\gamma_j(\ell) = c\,(1+j)^{-(1+\alpha)/2}\sum_{\ell\in Z}(1+j+|\ell|)^{-(1+\alpha)/2} \le$$

$$\le 2c\,(1+j)^{-(1+\alpha)/2}\int_0^\infty(1+j+t)^{-(1+\alpha)/2}dt = c\,(1+j)^{-\alpha}, \quad \alpha>1,$$

and Lemma 3.2 yields

$$\|\sum_{|k|\le N}Q^j_{k,N}f\|_2 = \|\sum_{|k|\le N}r_{k+N+1}(t)f_{j,k}\|_2 \le c\,(1+j)^{-\alpha}\|f\|_2,$$

c being independent of $r_{k+N+1}(t)$ and N. This implies (3.8).

Let us prove (3.9). By Plancherel's Theorem and the definition of Φ_ℓ ($\hat\Phi_\ell(\xi) = 1$ for $2^\ell|\xi| \le 2$),

$$\|(Q^j_{i,N})^* Q^j_{k,N} f\|_2^2 \le \int_{R^n} |1 - \hat\Phi_k(\xi)|^2 |1 - \hat\Phi_i(\xi)|^2 |\hat\omega_{j+k}(\xi)\overline{\hat\omega_{j+i}}(\xi)|^2 |\hat f(\xi)|^2 d\xi \le$$

$$\le \int_{|\xi| > 2^{1-\min(i,k)}} |\hat\omega_{j+k}(\xi)\overline{\hat\omega_{j+i}}(\xi)|^2 |\hat f(\xi)|^2 d\xi.$$

By Lemma 2.4 the last integral does not exceed

$$\int_{|\xi| > 2^{1-\min(i,k)}} |\hat f(\xi)|^2 [\log(2^{j+k}|\xi|) \log(2^{j+i}|\xi|)]^{-2-2\alpha} d\xi \le c^2 \, a^j_{i,k} \, \|\hat f\|_2^2,$$

where $a^j_{i,k} = [(j+k+1-\min(i,k))(j+i+1-\min(i,k))]^{-2-2\alpha} = [(j+1)(j+1+|k-i|)]^{-2-2\alpha}$, and (3.9) follows. Λ

Corollary 3.4. *Under the conditions of Lemma 3.3 (ii),*

$$\|Q_j f\|_p \le c\,(1+j)^{-\alpha\lambda} \|f\|_p, \quad j \ge 0, \quad f \in L^p, \tag{3.10}$$

where $\lambda/2 < 1/p < 1 - \lambda/2$ *if* $0 \le \lambda < 1$, $p = 2$ *if* $\lambda = 1$ *, and* c *is independent of* j.

Proof: Since Q_j is not a linear operator we cannot interpolate between (3.6) and (3.7) directly. Therefore we proceed as in [19], p. 280–281 (see also [8, p. 60]). Redenote $f_{j,k} = Q_{j,k} f$ so that $Q_j f = \sup_k |Q_{j,k} f|$ (cf. (3.5)). Let $\mathcal{K}$ be the set of all measurable integer-valued functions $k(x)$ on R^n. Given $k(x) \in \mathcal{K}$, define a linear operator

$$Q_{j,k(x)} f(x) = c_\nu \Bigg[\int_{2^{j+k(x)}}^{2^{j+k(x)+1}} \frac{dr}{r} \int_{\Sigma_{n-1}} f(x - r\theta) d\nu(\theta) -$$
$$- \int_{R^n} f(x-y) dy \int_{2^{j+k(x)}}^{2^{j+k(x)+1}} \frac{dr}{r} \int_{\Sigma_{n-1}} 2^{-nk(x)} \Phi(2^{-k(x)}(y - r\theta)) d\nu(\theta) \Bigg],$$

so that

$$\sup_{k \in \mathcal{K}} |Q_{j,k(x)} f(x)| = Q_j f(x). \tag{3.11}$$

By (3.6), for $f \in L^q$ we have

$$\| \sup_{k\in\mathcal{K}} |Q_{j,k(x)} f| \, \|_q = \|Q_j f\|_q \le c \, \|f\|_q \quad \forall q \in (1,\infty). \tag{3.12}$$

Moreover,

$$\sup_{k\in\mathcal{K}} \|Q_{j,k(x)} f\|_q = \|Q_j f\|_q. \tag{3.13}$$

Indeed, by (3.11) there is a sequence $\{k_\ell(x)\} \subset \mathcal{K}$ such that $\lim\limits_{\ell\to\infty} |Q_{j,k_\ell(x)} f(x)| = Q_j f(x)$, and therefore $\lim\limits_{\ell\to\infty} \|Q_{j,k_\ell(x)} f\|_q = \|Q_j f\|_q$ (use the Lebesgue theorem on dominated convergence together with (3.12)). The last equality implies (3.13) because $\|Q_{j,k(x)} f\|_q \le \|Q_j f\|_q \quad \forall k \in \mathcal{K}$.

Since (3.6) and (3.7) are valid for $Q_{j,k(x)}$, then

$$\|Q_{j,k(x)} f\|_p \le c \, (1+j)^{-\alpha\lambda} \|f\|_p, \tag{3.14}$$

where c is independent of j and $k(x), \lambda$ and p are as required. The relations (3.14) and (3.13) imply (3.10). Λ

4. Proof of Theorem B.

Step 1. Let us prove (1.8) for $f \in C_c^\infty(\mathrm{R}^n)$. Suppose that $2^{j-1} \le \varepsilon < 2^j$, $2^{\ell-1} \le \rho < 2^\ell$ for some $j, \ell \in \mathrm{Z}$. Then

$$T_\nu^{\varepsilon,\rho} f = T_\nu^{\varepsilon,\infty} f - T_\nu^{\rho,\infty} f, \tag{4.1}$$

$$T_\nu^{\varepsilon,\infty} f(x) = \sum_{k=j}^\infty (\omega_k * f)(x) + \int_\varepsilon^{2^j} \frac{dr}{r} \int_{\Sigma_{n-1}} f(x - r\theta) d\nu(\theta),$$

and

$$\sup_{0<\varepsilon<\rho<\infty} |T_\nu^{\varepsilon,\rho} f(x)| \le 2 \sup_{j\in\mathrm{Z}} \Big| \sum_{k=j}^\infty (\omega_k * f)(x) \Big| + 2 \sup_{j\in\mathrm{Z}} (|\omega_j| * |f|)(x). \tag{4.2}$$

By Lemma 2.3 (with $s = p$),

$$\| \sup_{j\in\mathrm{Z}} (|\omega_j| * |f|) \, \|_p \le c \, \|f\|_p \quad \forall p \in (1,\infty). \tag{4.3}$$

Let us estimate the first term in the right-hand side of (4.2). We take $\Phi_j(x) = 2^{-jn}\Phi(2^{-j}x)$ with Φ as in Lemma 3.3 (ii). Then

$$\sup_{j\in Z} |\sum_{k=j}^{\infty} \omega_k * f| \leq \sup_{j\in Z} \Big|(\delta - \Phi_j) * \Big(\sum_{k=j}^{\infty} \omega_k * f\Big)\Big| + \sup_{j\in Z} \Big|\Phi_j * \Big(\sum_{k=j}^{\infty} \omega_k * f\Big)\Big|, \tag{4.4}$$

δ being the Dirac delta function. By (3.5) and (3.10),

$$\Big\| \sup_{j\in Z} \Big|(\delta - \Phi_j) * \Big(\sum_{k=j}^{\infty} \omega_k * f\Big)\Big| \Big\|_p \leq \| \sum_{j=0}^{\infty} Q_j f\|_p \leq c\,\|f\|_p, \tag{4.5}$$

provided (1.9). Furthermore, by Theorem A and Lemma 3.1 (choose $\sigma_k = \omega_k$),

$$\Big\| \sup_{j\in Z} \Big|\Phi_j * \Big(\sum_{k=j}^{\infty} \omega_k * f\Big)\Big| \Big\|_p \leq c\,\|f\|_p. \tag{4.6}$$

These estimates imply (1.8).

Step 2. Suppose that $f \in L^p$, $\tilde{T}_\nu : L^p \to L^p$ is an extension of the operator T_ν, the existence of which was stated in Theorem A. Let us prove that $\Delta = \|T_\nu^{\varepsilon,\rho} f - \tilde{T}_\nu f\|_p \to 0$ as $\varepsilon \to 0,\ \rho \to \infty$ for p satisfying (1.9). This result is a consequence of the uniform estimate

$$\sup_{0<\varepsilon<\rho<\infty} \|T_\nu^{\varepsilon,\rho} f\|_p \leq A\,\|f\|_p, \quad A = \text{const}. \tag{4.7}$$

Indeed, if $\{f_m\} \subset C_c^\infty$, $\lim\limits_{m\to\infty} \|f - f_m\|_p = 0$, then

$$\begin{aligned}\Delta \leq\ & \|T_\nu^{\varepsilon,\rho}(f - f_m)\|_p + \|T_\nu^{\varepsilon,\rho} f_m - \tilde{T}_\nu f_m\|_p + \|\tilde{T}_\nu(f_m - f)\|_p \leq \\ & \leq A\|f - f_m\|_p + \|T_\nu^{\varepsilon,\rho} f_m - \tilde{T}_\nu f_m\|_p + c\,\|f_m - f\|_p.\end{aligned}$$

The first and the last terms become small due to m, the second term tends to 0 as $\varepsilon \to 0$ and $\rho \to \infty$ by the Lebesgue theorem of dominated convergence which is applicable owing to Step 1.

In order to prove (4.7) we note that the uniform inequality $\|T_\nu^{\varepsilon,\rho}\omega\|_p \leq A\|\omega\|_p$ holds for $\omega \in C_c^\infty$ due to Step 1. Hence it can be extended to all $f \in L^p$, and we get $\|T_\nu^{\varepsilon,\rho} f\|_p \leq A\|f\|_p$. This gives (4.7).

Step 3. Let us prove (1.8) for $f \in L^p$. It suffices to check (4.1)-(4.6) for such an f. By the reasons, which are similar to [1], p. 292, the integral $T_{|\nu|}^{\varepsilon,\infty}|f|(x)$ is well-defined for a.e.

$x \in \mathrm{R}^n$, which implies the a.e. convergence of the series $A_j f(x) = \sum\limits_{k=j}^{\infty} \omega_k * f(x)$ for each $j \in \mathrm{Z}$. This gives (4.1)–(4.4). The validity of (4.5) follows from Corollary 3.4. Let us check (4.6). Note that the relation $\Delta \to 0$ in Step 2 implies $\tilde{T}_\nu f = \underset{\substack{m\to\infty \\ \ell\to-\infty}}{\overset{(L^p)}{\lim}} \sum\limits_{k=\ell}^{m} \omega_k * f$. Suppose for a moment that the series $A_j f$ and $B_j f = \sum\limits_{k=-\infty}^{j-1} \omega_k * f$ converge in the L^p-norm. Then $A_j f \overset{a.e.}{=} \tilde{T}_\nu f - B_j f$, and the left-hand side of (4.6) does not exceed

$$\| \sup_{j\in \mathrm{Z}} |\Phi_j * \tilde{T}_\nu f| \, \|_p + \| \sup_{j\in \mathrm{Z}} |\Phi_j * B_j f| \, \|_p, \tag{4.8}$$

(the series $A_j f(x)$ being also convergent in the a. e. sense). By [17, p. 27] and Step 2, we can estimate the first term in (4.8):

$$\| \sup_{j\in \mathrm{Z}} |\Phi_j * \tilde{T}_\nu f| \, \|_p \le c \, \|M(\tilde{T}_\nu f)\|_p \le c \, \|\tilde{T}_\nu f\|_p \le c \, \|f\|_p, \tag{4.9}$$

where M is the Hardy-Littlewood maximal operator.

Let us estimate the second term in (4.8). By the reasons, which are similar to [19], p. 162-163, the series $B_j \Phi_j(x)$ converges for each x. Furthermore, by (3.2) (with $\sigma_k = \omega_k$), $|B_j \Phi_j(x)| \le c \, \psi_j(x)$, $\quad \psi_j(x) = 2^{-jn}/(1+|2^{-j}x|)^{n+1}$. Hence $\Phi_j * B_j f \overset{a.e.}{=} B_j \Phi_j * f$ (both functions belong to L^p and coincide in the weak sense), and we obtain

$$\| \sup_{j\in \mathrm{Z}} |\Phi_j * B_j f| \, \|_p \le c \, \| \sup_{j\in \mathrm{Z}} (|\psi_j| * |f|) \, \|_p \le c \, \|Mf\|_p \le c \, \|f\|_p. \tag{4.10}$$

By (4.8)–(4.10) we get (4.6).

It remains to check the L^p-convergence of the series $A_j f$ and $B_j f$. The operators A_j and B_j extend as L^p-bounded operators with the norms, independent of j. To see this, one should use Lemmas 2.2 and 2.3 by putting $\sigma_k = \begin{cases} \omega_k & \text{if } k \ge j \\ 0 & \text{if } k < j \end{cases}$ for A_j, and $\sigma_k = \begin{cases} 0 & \text{if } k \ge j \\ \omega_k & \text{if } k < j \end{cases}$ for B_j. By Step 1, for $f \in C_c^\infty$ we have

$$\| \sup_{m\in \mathrm{Z}} |\sum_{k=j}^{m} \omega_k * f| \, \|_p \le \| \sup_{\ell,m} |\sum_{k=\ell}^{m} \omega_k * f| \, \|_p \le \| \sup_{0<\varepsilon<\rho<\infty} |T_\nu^{\varepsilon,\rho} f| \, \|_p \le c \, \|f\|_p.$$

Hence, by the reasons, which are similar to those in Step 2, we obtain an L^p-convergence of $A_j f$ and $B_j f$ for $f \in L^p$.

Thus, the maximal estimate (1.8) is proved. The a.e. convergence of $T_\nu^{\varepsilon,\rho} f$ then follows in a standard way (use, e.g., Theorem 3.12 from [19, Chapter II]). Λ

5. Singular integrals, generated by zonal spherical measures.

5.1 Auxiliary results.

Lemma 5.1. *Suppose that $n>2$, Λ is an $SO(n-1)$-invariant subset of Σ_{n-1}, $\nu\in M_z(\Sigma_{n-1})$ (see Notation). If $f\in L^1(\Sigma_{n-1},d\nu)$, then*

$$\int\limits_{\Lambda} f(\vartheta)d\nu(\vartheta)=\frac{1}{\sigma_{n-2}}\int\limits_{\Lambda} d\nu(\vartheta)\int\limits_{\Sigma_{n-2}} f(\sqrt{1-\vartheta_n^2}\sigma+\vartheta_n e_n)d\sigma, \tag{5.1}$$

where $\sigma_{n-2}=|\Sigma_{n-1}|$ and $d\sigma$ is the usual Lebesgue measure on Σ_{n-2}.

Proof. Let $\vartheta=(\sin\theta)\sigma+(\cos\theta)e_n,\ \sigma\in\Sigma_{n-2},\ \cos\theta=\vartheta_n$. Then

$$\int\limits_{\Lambda} f(\vartheta)d\nu(\vartheta)=\int\limits_{SO(n-1)} d\gamma\int\limits_{\Lambda} f(\gamma\vartheta)d\nu(\vartheta)=\int\limits_{\Lambda} d\nu(\vartheta)\int\limits_{SO(n-1)} f((\sin\theta)\gamma\sigma+(\cos\theta)e_n)d\gamma=$$

$$=\frac{1}{\sigma_{n-2}}\int\limits_{\Lambda} d\nu(\vartheta)\int\limits_{\Sigma_{n-2}} f((\sin\theta)\sigma+(\cos\theta)e_n)d\sigma,$$

which gives (5.1). Λ

Lemma 5.2. *Let ν and Λ be the same as in Lemma 5.1. Then for $\beta\in(0,1/2)$ and $n>2$,*

$$\sup_{|\xi|=1}\int\limits_{\Lambda}|\theta\cdot\xi|^{-\beta}d|\nu|(\theta)\le c\int\limits_{\Lambda}|\theta_n|^{-\beta}(1-\theta_n^2)^{-\beta/2}d|\nu|(\theta),\qquad c=c(n,\beta). \tag{5.2}$$

Proof. Let $\xi=(\tilde\xi,\xi_n),\ \tilde\xi\in\mathrm{R}^{n-1}$. By (5.1),

$$\int\limits_{\Lambda}|\theta\cdot\xi|^{-\beta}d|\nu|(\theta)=\frac{1}{\sigma_{n-2}}\int\limits_{\Lambda}d|\nu|(\theta)\int\limits_{\Sigma_{n-2}}|\sqrt{1-\theta_n^2}\;\sigma\cdot\tilde\xi+\theta_n\xi_n|^{-\beta}d\sigma\;= \tag{5.3}$$

$$=\frac{\sigma_{n-3}}{\sigma_{n-2}}\int\limits_{\Lambda}A(\xi,\theta)d|\nu|(\theta),\quad A(\xi,\theta)=\int\limits_{-1}^{1}|t\sqrt{(1-\theta_n^2)(1-\xi_n^2)}+\theta_n\xi_n|^{-\beta}(1-t^2)^{n/2-2}dt.$$

If $|\theta_n|\ge\sqrt{1-\xi_n^2}$, then $|\xi_n|\ge\sqrt{1-\theta_n^2},\ b\stackrel{\text{def}}{=}\sqrt{(1-\theta_n^2)(1-\xi_n^2)}/|\theta_n\xi_n|\le1$, and we have

$$A(\xi,\theta)\le\int\limits_{-1}^{1}\frac{(1-t^2)^{n/2-2}dt}{(|\theta_n\xi_n|-|t|\sqrt{(1-\theta_n^2)(1-\xi_n^2)})^{\beta}}\le|\theta_n\xi_n|^{-\beta}\int\limits_{-1}^{1}\frac{(1-t^2)^{n/2-2}dt}{(1-b|t|)^{\beta}}\le$$

$$\le2\,|\theta_n|^{-\beta}(1-\theta_n^2)^{-\beta/2}\int\limits_{0}^{1}\frac{(1-t^2)^{n/2-2}dt}{(1-t)^{\beta}}=c\,|\theta_n|^{-\beta}(1-\theta_n^2)^{-\beta/2},\quad c=\text{const}.$$

If $|\theta_n| < \sqrt{1-\xi_n^2}$, i.e. $|\xi_n| < \sqrt{1-\theta_n^2}$, then $a \stackrel{\text{def}}{=} -\theta_n\xi_n/\sqrt{(1-\xi_n^2)(1-\theta_n^2)} \in (-1,1)$, and we get

$$A(\xi,\theta) \le [(1-\theta_n^2)(1-\xi_n^2)]^{-\beta/2} \int_{-1}^{1} \frac{(1-t^2)^{n/2-2}dt}{|t-a|^\beta} \le$$

$$\le |\theta_n|^{-\beta}(1-\theta_n^2)^{-\beta/2}[I(a)+I(-a)], \quad I(a) = \int_{-1}^{a} \frac{(1-t^2)^{n/2-2}dt}{(a-t)^\beta}. \tag{5.4}$$

By the formulas 2.2.6.1 from [10], and 9.102.2 from [5] we obtain

$$I(a) = \frac{2^{n/2-2}B(n/2-1,1-\beta)}{(a+1)^{1+\beta-n/2}} F\Big(\frac{n}{2}-1, 2-\frac{n}{2}; \frac{n}{2}-\beta; \frac{a+1}{2}\Big) \le c(n,\beta) < \infty,$$

$0<\beta<1/2$. The same estimate holds for $I(-a)$. Λ

Lemma 5.3. *Let $\nu \in M_z(\Sigma_{n-1})$, $n>2$. Then*

$$\sup_{|\xi|=1} \int_{\Sigma_{n-1}} \log\frac{1}{|\theta\cdot\xi|} d|\nu|(\theta) < \infty, \tag{5.5}$$

if and only if

$$\int_{\Sigma_{n-1}} \log\frac{1}{|\theta_n|\sqrt{1-\theta_n^2}} d|\nu|(\theta) < \infty. \tag{5.6}$$

Proof. Denote

$$R(\xi_n,\theta_n) = \int_{-1}^{1} (1-t^2)^{n/2-2} \log\frac{1}{|t\sqrt{(1-\theta_n^2)(1-\xi_n^2)}+\theta_n\xi_n|} dt.$$

As in (5.3) we have

$$\int_{\Sigma_{n-1}} \log\frac{1}{|\theta\cdot\xi|} d|\nu|(\theta) = \frac{\sigma_{n-3}}{\sigma_{n-2}} \int_{\Sigma_{n-1}} R(\xi_n,\theta_n)d|\nu|(\theta) \stackrel{\text{def}}{=} \frac{\sigma_{n-3}}{\sigma_{n-2}} K(\xi_n). \tag{5.7}$$

Using the same notation as in the proof of Lemma 5.2 we have for $|\theta_n| \ge \sqrt{1-\xi_n^2}$:

$$R(\xi_n,\theta_n) \le \int_{-1}^{1} (1-t^2)^{n/2-2} \log \frac{1}{|\theta_n\xi_n|(1-|t|b)} dt \le$$

$$\le c_1 \log \frac{1}{|\theta_n\xi_n|} + 2\int_0^1 (1-t^2)^{n/2-2} \log \frac{1}{1-t} dt \le c_1 \log \frac{1}{|\theta_n|\sqrt{1-\theta_n^2}} + c_2;$$

for $|\theta_n| < \sqrt{1-\xi_n^2}$:

$$R(\xi_n,\theta_n) \le \log \frac{1}{\sqrt{(1-\theta_n^2)(1-\xi_n^2)}} \Big[c_1 + \int_{-1}^{1} (1-t^2)^{n/2-2} \log \frac{1}{|t-a|} dt \Big] <$$

$$< \log \frac{1}{|\theta_n|\sqrt{1-\theta_n^2}} \Big[c_1 + c_2 \int_{-1}^{1} \frac{(1-t^2)^{n/2-2}}{|t-a|^{1/4}} dt \Big] \le c \log \frac{1}{|\theta_n|\sqrt{1-\theta_n^2}},$$

c being independent of a (see the estimate of the integral in (5.4)). Hence (5.6) implies (5.5). Conversely, if (5.5) holds, then (see (5.7)) $K(0) < \infty$ and $K(\pm 1) < \infty$. Since

$$K(0) = \int_{\Sigma_{n-1}} \log \frac{1}{\sqrt{1-\theta_n^2}} d|\nu|(\theta) \int_{-1}^{1} (1-t^2)^{n/2-2} dt + \int_{\Sigma_{n-1}} d|\nu|(\theta) \int_{-1}^{1} (1-t^2)^{n/2-2} \log \frac{1}{|t|} dt,$$

and

$$K(\pm 1) = \int_{\Sigma_{n-1}} \log \frac{1}{|\theta_n|} d|\nu|(\theta) \int_{-1}^{1} (1-t^2)^{n/2-2} dt,$$

then

$$\int_{\Sigma_{n-1}} \log \frac{1}{\sqrt{1-\theta_n^2}} d|\nu|(\theta) < \infty, \qquad \int_{\Sigma_{n-1}} \log \frac{1}{|\theta_n|} d|\nu|(\theta) < \infty,$$

and (5.6) follows. Λ

The next result will be used in the proof of Theorem C.

Theorem 5.4 (cf. [20], p. 3). *Let $\{\sigma_j\}_{j\in \mathrm{Z}}$ be a sequence of finite Borel measures on* R^n, *which for integers $m \ge 0$ admit a splitting $\sigma_j = U_j^m + L_j^m$ into Borel measures U_j^m and L_j^m so that*

(5.8) $\qquad U_j^m$ *and* L_j^m *are supported in* $\{x : |x| < c\, 2^j\}$;

(5.9) $$\|L_j^m\| \le c, \qquad |\hat{L}_j^m(\xi)| \le \frac{c\, 2^{am}}{(2^j|\xi|)^\alpha}, \quad \alpha > 0;$$

(5.10) $$\sup_j \sum_{m=0}^{\infty} \|U_j^m\| \le c.$$

Here c and a are nonnegative constants, independent of m and j. If the operator

(5.11) $$Tf = \sum_{j=-\infty}^{\infty} \sigma_j * f, \qquad f \in C_c^\infty(\mathrm{R}^n),$$

extends to a bounded operator on $L^2(\mathrm{R}^n)$, then T extends to a bounded operator on $L^p(\mathrm{R}^n)$, $1 < p < \infty$.

5.2. Proof of Theorem C and Corollary 1.4.

We denote

(5.12) $$\Gamma_m = \{\theta \in \Sigma_{n-1} : |\theta_n|\sqrt{1-\theta_n^2} < 2^{-m}\}, \quad \Gamma_m^c = \Sigma_{n-1} \setminus \Gamma_m,$$

and set $\sigma_j = U_j^m + L_j^m$, where the measures L_j^m and U_j^m are defined by

(5.13) $$(L_j^m, g) = c_\nu \int_{2^j}^{2^{j+1}} \frac{dr}{r} \int_{\Gamma_m^c} g(r\theta) d\nu(\theta), \qquad c_\nu = \frac{1}{\|\nu\| \log 2};$$

(5.14) $$(U_j^m, g) = c_\nu \int_{2^j}^{2^{j+1}} \frac{dr}{r} \int_{\Gamma_m} g(r\theta) d\nu(\theta),$$

$g \in C_0(\mathrm{R}^n)$. Suppose that $f \in C_c^\infty(\mathrm{R}^n)$. Then the series $T_\nu f(x) = \sum_{j\in\mathrm{Z}} (\sigma_j * f)(x)$, converges for each $x \in \mathrm{R}^n$. By Lemma 5.3, and by the reasons, which are similar to [18, p. 40], T_ν extends to a bounded operator on $L^2(\mathrm{R}^n)$.

Thus, by Theorem 5.4, it suffices to check (5.8)–(5.10). The validity of (5.8) and the first condition in (5.9) is clear. To check the second inequality in (5.9), we note that

$$(L_j^m)^\wedge(\xi) = c_\nu \int_{\Gamma_m^c} d\nu(\theta) \Big\{ \int_{2^j}^{2^{j+1}} e^{-2\pi i r\theta\cdot\xi} \frac{dr}{r} \Big\}.$$

The integral in brackets is dominated by $\log 2$ and also by $2^{-j}|\theta\cdot\xi|^{-1}$. Hence, it does not exceed $c\,(2^{-j}|\theta\cdot\xi|)^{-\alpha}$ for any $\alpha\in(0,1)$. Let $\alpha=1/4$, $\xi'=\xi/|\xi|$. Then by Lemma 5.2,

$$|(L_j^m)^\wedge(\xi)|\le\frac{c}{(2^j|\xi|)^{1/4}}\int\limits_{\Gamma_m^c}\frac{d|\nu|(\theta)}{|\theta\cdot\xi'|^{1/4}}\le$$

$$\le\frac{c}{(2^j|\xi|)^{1/4}}\int\limits_{\Gamma_m^c}\frac{d|\nu|(\theta)}{(|\theta_n|\sqrt{1-\theta_n^2})^{1/4}}\le\frac{c\,2^{m/4}}{(2^j|\xi|)^{1/4}}.$$

It remains to check (5.10). By (5.14),

$$\|U_j^m\|\le c_\nu\int\limits_{2^j}^{2^{j+1}}\frac{dr}{r}\int\limits_{\Gamma_m}d|\nu|(\theta)\le c\int\limits_{\Gamma_m}d|\nu|(\theta)$$

(see the proof of (2.11)). Hence (see (5.12))

$$\sum_{m=0}^{\infty}\|U_j^m\|\le c\sum_{m=0}^{\infty}\int\limits_{\Gamma_m}d|\nu|(\theta)=c\int\limits_{\Sigma_{n-1}}d|\nu|(\theta)\Big[\sum_{m<\log_2(1/|\theta_n|\sqrt{1-\theta_n^2})}1\Big]\le$$

$$\le c\int\limits_{\Sigma_{n-1}}\log\frac{1}{|\theta_n|\sqrt{1-\theta_n^2}}d|\nu|(\theta)<\infty,$$

which gives (5.10). The statement (a) is proved. The statement (b) follows from Theorem B by taking into account that (1.6) holds for all $\alpha>0$ owing to (1.11) and Lemma 5.2. Λ

Corollary 1.4 is a consequence of part (a) of Theorem C and Lemma 5.3.

5.3. Proof of Corollary 1.5.

The required function can be constructed as follows. Denote

$$\Lambda_1=\{\theta\in\Sigma_{n-1}:1/4<\theta_n<3/4\},\quad\Lambda_2=\{\theta\in\Sigma_{n-1}:1/3\le\theta_n\le1/2\}(\subset\Lambda_1),$$

and let $\psi:\Sigma_{n-1}\to R$ be an integrable even zonal function such that $\psi(\theta)>0$ on Λ_1 and $|\psi|\log(1+|\psi|)\notin L^1(\Lambda_2)$. We define

$$\Omega(\theta)=\begin{cases}\lambda\,|\theta_n|^{-1}(\log|\theta_n|^{-1})^{-2}(\log\log|\theta_n|^{-1})^{-2} & \text{if } |\theta_n|<1/100,\\ \psi(\theta)\quad\text{if }\ 1/4<|\theta_n|<3/4, & \\ 0\quad\text{otherwise,} & \end{cases}$$

where $\lambda(<0)$ is choosen so that $\int_{\Sigma_{n-1}} \Omega(\theta)d\theta = 0$. For all $\alpha > 0$,

$$\sup_{|\xi|=1} \int_{\Sigma_{n-1}} \left(\log \frac{1}{|\xi\cdot\theta|}\right)^{1+\alpha} |\Omega(\theta)|d\theta \geq \int_{\{\theta:|\theta_n|<1/100\}} \left(\log \frac{1}{|\theta_n|}\right)^{1+\alpha} |\Omega(\theta)|d\theta = \infty,$$

i.e. (1.4) fails. Let us prove that $\Omega \notin H^1(\Sigma_{n-1})$. Assuming the contrary and setting $g(x) = |x|^{1-n}\Omega(x'),\ x' = x/|x|$, if $|x| \leq 2$, and $g(x) \equiv 0$ if $|x| > 2$, we obtain $g(x) \in H^1(\mathrm{R}^n)$ (see Lemma 2.5 from [15]). Since g is positive on the open set

$$\tilde{\Lambda}_1 = \{x = r\theta \in \mathrm{R}^n : 1/4 < r < 2,\ \theta \in \Lambda_1\},$$

then g belongs to the class $L\log L$ on any compact $K \subset \tilde{\Lambda}_1$ ([17], p. 128). By choosing $K = \{x = r\theta \in \mathrm{R}^n : 1/2 \leq r \leq 1,\ \theta \in \Lambda_2\}$ we get

$$\infty > \int_K g(x)\log(1+g(x))dx = \int_{1/2}^{1} dr \int_{\Lambda_2} \Omega(x')\log\left(1+\frac{\Omega(x')}{r^{n-1}}\right)dx' \geq$$
$$\geq \frac{1}{2}\int_{\Lambda_2} \psi(x')\log(1+\psi(x'))dx' = \infty$$

due to the choice of ψ. This contradiction shows that $\Omega \notin H^1(\Sigma_{n-1})$.

It remains to note that for $d\nu(\theta) = \Omega(\theta)d\theta$ the operator T_ν extends to a bounded operator on $L^p(\mathrm{R}^n)$ $\forall p \in (1,\infty)$ according to Corollary 1.4 and Lemma 5.3.

5.4. Proof of Corollary 1.6.

Let Λ_1, Λ_2 and ψ be the same as in the previous subsection. Consider the function

$$\Omega(\theta) = \begin{cases} \lambda & \text{if } 1/5 < |\theta_n| \leq 1/4, \\ \psi(\theta) & \text{if } 1/4 < |\theta_n| < 3/4, \\ 0, & \text{otherwise}, \end{cases} \tag{5.15}$$

where $\lambda < 0$ is such that $\int_{\Sigma_{n-1}} \Omega(\theta)d\theta = 0$. By Lemma 5.2 the function (5.15) satisfies (1.4) for all $\alpha > 0$. On the other hand, $\Omega \notin H^1(\Sigma_{n-1})$ (see the proof of Corollary 1.5). Λ

Proof of Proposition 1.7.

Let $x = (x_1, \tilde{x}) \in \mathrm{R}^n$, $\tilde{x} = (x_2, \ldots, x_n) \in \mathrm{R}^{n-1}$. We set $\Omega(x') = \int_0^\infty r^{n-1}g(rx')dr$, $x' = x/|x|$, where $g(x) = u(x_1)v(\tilde{x})$,

$$u(x_1) = \begin{cases} {x_1}^{-1}(\log|x_1|^{-1})^{-1-\varepsilon} & \text{if} \quad 0 < |x_1| < 1/100, \\ 0 & \text{if} \quad |x_1| > 1/100, \end{cases} \qquad 0 < \varepsilon < 1,$$

$$v(\tilde{x}) = \begin{cases} sgn\ x_2 & \text{if} \quad 1/100 < |\tilde{x}| < 2/100, \\ 0 & \text{otherwise.} \end{cases}$$

It is clear that $\int_{\Sigma_{n-1}} \Omega(x')dx' = \int_{\mathrm{R}^n} g(x)dx = 0$. Let us check that $\Omega \in H^1(\Sigma_{n-1})$. By Lemma 2.4 from [15], it suffices to show that $g \in H^1(\mathrm{R}^n)$.

Since $u \in H^1(\mathrm{R})$ (see Section 6.2 of [17], p. 178), and $v \in H^1(\mathrm{R}^{n-1})$ (see Section 1.2.4. of [17], p. 92), there are Schwartz functions $\Phi_1(x_1)$ and $\Phi_2(\tilde{x})$ with nonvanishing integrals such that

$$\sup_{t>0} |((\Phi_1)_t * u)(x_1)| \in L^1(\mathrm{R}), \qquad \sup_{t>0} |((\Phi_2)_t * v)(\tilde{x})| \in L^1(\mathrm{R}^{n-1}) \tag{6.1}$$

(cf. Theorem 1 from [17], p. 91). In view of (6.1), $\sup_{t>0} |(\Phi_t * g)(x)| \in L^1(\mathrm{R}^n)$, where $\Phi(x) = \Phi_1(x_1)\Phi_2(\tilde{x})$ is a Schwartz function. This gives $g \in H^1(\mathrm{R}^n)$.

Let us check (1.12). We set $a(\xi) = \int\limits_{\Sigma_{n-1}} |\Omega(x')| \log(1/|\xi \cdot x'|)\, dx'$. Since $g(x) = |g(x)|\, sgn\ x_1\ sgn\ x_2$, then $|\Omega(x')| = \int_0^\infty r^{n-1}|g(rx')|dr$. Hence

$$\sup_{|\xi|=1} a(\xi) \geq a(e_1) = \int\limits_{\mathrm{R}^n} |g(x)| \log \frac{1}{|x_1'|} dx = \int\limits_{\mathrm{R}^n} |g(x)| \log \frac{1}{|x_1|} dx - \int\limits_{\mathrm{R}^n} |g(x)| \log \frac{1}{|x|} dx \geq \infty$$

because the first integral in (6.2) is infinite and the second one does not exceed

$$\int\limits_{\mathrm{R}} |u(x_1)|dx_1 \int\limits_{\mathrm{R}^{n-1}} |v(\tilde{x})| \log \frac{1}{|\tilde{x}|} d\tilde{x} < \infty.$$

Thus we are done. Λ

7. Examples.

Below we give examples of singular non-zonal measures, which satisfy (1.6) for all $\alpha > 0$. For these measures all statements of Theorem B hold in the maximal range $1 < p < \infty$.

Example 7.1 ($n = 2$). Consider the distribution function $C(x)$ of the middle third Cantor set on [0,1] (see [14], p. 145). Let $C_{2\pi}(x) = 2\pi C(x/2\pi)$, so that $C_{2\pi}(0) = 0$ and $C_{2\pi}(2\pi) = 2\pi$. By setting

$$g(x) = \begin{cases} x, & x \in [0, 2\pi/6], \\ 2\pi/3 - x, & x \in [2\pi/6, 2\pi/3], \\ 0, & x \in [2\pi/3, 4\pi/3], \\ 4\pi/3 - x, & x \in [4\pi/3, 10\pi/6], \\ x - 2\pi, & x \in [10\pi/6, 2\pi], \end{cases}$$

we define an auxiliary measure σ on $[0, 2\pi]$ by

$$\sigma(E) = \int_E d\psi(x), \quad \psi(x) = (g \circ C_{2\pi})(x),$$

E being a Borel subset of $[0, 2\pi]$ (since ψ is a function of bounded variation, this definition is correct). Let $h : [0, 2\pi] \to \Sigma_1$ be a canonical map so that $h(\theta) = (\cos\theta, \sin\theta) \in \Sigma_1$. We define the required measure ν on Σ_1 as an image of σ under the mapping h. It is clear that $\nu(\Sigma_1) = \sigma([0, 2\pi]) = 0$. Moreover, one can readily check that the total variation $\overset{x}{\underset{0}{V}}\psi$ of ψ on $[0, x]$ coincides with $C(x)$. Hence for any interval $[a, b] \subset [0, 2\pi]$, we have

$$|\sigma|([a, b]) \overset{\text{def}}{=} \overset{b}{\underset{a}{V}}\psi = \overset{b}{\underset{0}{V}}\psi - \overset{a}{\underset{0}{V}}\psi = C(b) - C(a),$$

and therefore [14, p. 157]

$$|\sigma|([a, b]) \le c\,(b - a)^{\log_3 2}. \tag{7.1}$$

Let us show that

$$\sup_{|\xi|=1} \int_{\Sigma_1} |\vartheta \cdot \xi|^{-\beta} d|\nu|(\vartheta) < \infty \tag{7.2}$$

for all $\beta < \log_3 2$ (this implies (1.6) for all $\alpha > 0$). Fix $\xi = (\cos\varphi, \sin\varphi) \in \Sigma_1$, and $\varepsilon \in (0, 2^{-10})$. Suppose for a moment that

$$\int_{\Sigma_1} |\vartheta \cdot \xi|^{-\beta} d|\nu|(\vartheta) \le \int_0^{2\pi} |\cos(\varphi - \theta)|^{-\beta} d|\sigma|(\theta). \tag{7.3}$$

The right-hand side of (7.3) is equal to

$$\Big(\int_{A^1_\varepsilon(\varphi)} + \int_{A^2_\varepsilon(\varphi)} + \int_{A^3_\varepsilon(\varphi)} \Big) |\cos(\varphi - \theta)|^{-\beta} d|\sigma|(\theta) = I_1 + I_2 + I_3, \tag{7.4}$$

where

$$A^1_\varepsilon(\varphi) = \{\theta \in [0, 2\pi] : \pi/2 - \varepsilon < |\theta - \varphi| < \pi/2 + \varepsilon\},$$

$$A^2_\varepsilon(\varphi) = \{\theta \in [0, 2\pi] : 3\pi/2 - \varepsilon < |\theta - \varphi| < 3\pi/2 + \varepsilon\},$$

$$A^3_\varepsilon(\varphi) = [0, 2\pi] \setminus (A^1_\varepsilon(\varphi) \cup A^2_\varepsilon(\varphi)).$$

The third integral in (7.4) is dominated by $(\sin\varepsilon)^{-\beta}\ |\sigma|([0, 2\pi])$. Furthermore, by Theorem 1.15 from [9], p. 15,

$$I_1 \le c \int\limits_{A^1_\varepsilon(\varphi)} |\theta - \varphi - \frac{\pi}{2}|^{-\beta} d|\sigma|(\theta) = \beta(\int\limits_0^\varepsilon + \int\limits_\varepsilon^\infty)|\sigma|(\{\theta \in A^1_\varepsilon(\varphi) : |\theta - \varphi - \frac{\pi}{2}| \le t\})\frac{dt}{t^{\beta+1}}.$$

Both integrals are dominated by a constant which is independent of φ. For the second integral this is obvious. For the first one the statement holds due to estimate

$$|\sigma|(\{\theta \in A^1_\varepsilon(\varphi) : |\theta - \varphi - \pi/2| \le t\}) \le |\sigma|(\{\theta \in [0, 2\pi] : |\theta - \varphi - \pi/2| \le t\}) \le c\, t^{\log_3 2}$$

which follows from (7.1). For I_2 the argument is similar.

It remains to prove (7.3). For $N \in \mathrm{N}$ denote $D_\xi(N) = \{\vartheta \in \Sigma_1 : |\vartheta \cdot \xi|^{-\beta} \le N\}$, and let $\{S^\xi_m(\vartheta)\}^\infty_{m=1}$ be a sequence of simple functions, such that for each $\vartheta \in D_\xi(N)$, $0 \le S^\xi_1 \le \ldots \le S^\xi_m \le \ldots \le |\vartheta \cdot \xi|^{-\beta}$ and $S^\xi_m(\vartheta) \to |\vartheta \cdot \xi|^{-\beta}$ as $m \to \infty$. By the reasons, which are similar to those in the proof of (2.11), we have $|\nu|(E) \le |\sigma|(h^{-1}(E))\quad \forall E \in \mathcal{B}(\Sigma_1)$. Hence,

$$\int\limits_{D_\xi(N)} |\vartheta \cdot \xi|^{-\beta}\, d|\nu|(\vartheta) = \lim_{m\to\infty} \int\limits_{D_\xi(N)} S^\xi_m(\vartheta)\, d|\nu|(\vartheta) \le$$

$$\le \lim_{m\to\infty} \int\limits_{h^{-1}(D_\xi(N))} (S^\xi_m \circ h)(\theta)\, d|\sigma|(\theta) = \int\limits_{h^{-1}(D_\xi(N))} |\cos(\varphi - \theta)|^{-\beta} d|\sigma|(\theta),$$

where $h^{-1}(D_\xi(N)) = \{\theta \in [0, 2\pi] : |\cos(\varphi - \theta)|^{-\beta} \le N\}$. Tending N to infinity, we obtain (7.3).

The next example is motivated by Corollary 4.3 from [3], p. 553.

Example 7.2 $(n > 2)$. Define a measure ν on Σ_{n-1} by

$$\int\limits_{\Sigma_{n-1}} g(\vartheta)d\nu(\vartheta) = \int\limits_\Gamma g(y)\Omega(y)d_\Gamma y, \quad \int\limits_\Gamma \Omega(y)d_\Gamma y = 0,$$

where $\Gamma = \{\vartheta \in \Sigma_{n-1} : \vartheta_n = 1/2\}$, $\Omega \in L^q(\Gamma)$ for some $q > 1$, $g \in C(\Sigma_{n-1})$; $d_\Gamma y$ is the induced Lebesgue measure on Γ. By the reasons, which are similar to those in the proof of (7.3), and by Hölder's inequality we have

$$\int\limits_{\Sigma_{n-1}} |\xi \cdot \vartheta|^{-\beta} d|\nu|(\vartheta) \leq \int\limits_{\Gamma} |\xi \cdot y|^{-\beta} |\Omega(y)| d_\Gamma y \leq \; K^{1/p} \parallel \Omega \parallel_{L^q(\Gamma)},$$

where $1/p+1/q=1$ and $K = \int\limits_\Gamma |\xi \cdot y|^{-p\beta} d_\Gamma y$ is bounded uniformly in ξ for $0<\beta<1/2\,p$ (cf. Lemma 5.2). By Theorem A the relevant singular integral operator T_ν is bounded on L^p for all $1<p<\infty$.

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