Wavelet Type Representations in Fractional Calculus

By

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Abstract. Wavelet type representations of fractional integrals and derivatives are studied in the framework of L^p-spaces. These representations generalize the notion of Marchaud's fractional derivative and are intimately connected with the Calderón reproducing formula. By choosing a relevant "wavelet" measure we give a unified representation of the following basic objects in fractional calculus on the real line: the Riemann-Liouville fractional integrals, the Riesz potentials, the conjugate Riesz potentials, the inverses and linear combinations of these operators.

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Introduction

Fractional calculus is a well developed branch of real analysis with a rich history and a wide field of applications (see e.g. Samko, Kilbas and Marichev 1993, Gorenflo and Vessella 1991, Miller and Ross 1993). Fractional integrals have proved to be a convenient tool for studying function spaces of the fractional smoothness. Concerning the modern theory of function spaces the reader is referred to the books by H.Triebel (1983, 1992) where further references can be found. Numerous modifications of fractional integrals and derivatives (e.g, those of Riemann-Liouville, Weil, Marchaud, Riesz and many others) are known. Sometimes such modifications represent essentially different operators (cf. Riemann-Liouville fractional integrals, Riesz potentials and Bessel potentials). But sometimes some of them correspond to the same operator (on sufficiently good functions). The use of different constructions is caused in the last case by the wish to extend corresponding definition to wider classes of functions (cf., e.g., Liouville fractional derivatives and those of the Marchaud type).

In the present paper we introduce new representation for various fractional integrals and derivatives in the form

$$A^\alpha f = \frac{1}{c_\nu(\alpha)} \int_0^\infty \frac{f * \nu_t}{t^{1-\alpha}} dt, \quad \alpha \in \mathrm{C}, \tag{0.1}$$

where ν_t is a suitable dilated measure (or distribution), $c_\nu(\alpha)$ is a normalizing factor. Why is this integral so attractive? In the case $\alpha = 0$, A^α coincides with identity operator or with a Hilbert transform (depending on ν) and is intimately connected with Calderón's reproducing formula

$$f = \frac{1}{c_{u,v}} \int_0^\infty \frac{f * u_t * v_t}{t} dt. \tag{0.2}$$

(see e.g. Frazier, Jawerth and Weiss 1991, Rubin and Shamir 1995, Rubin 1995 and references therein). The operator (0.1) gives rise to natural generalization of (0.2) which we intend to investigate in this paper.

Apart form this, various fractional integrals (and derivatives) have the form (0.1). Really, by choosing $\nu = \delta_1$, (the Dirac unit mass at the point $x = 1$) and $c_\nu(\alpha) = \Gamma(\alpha)$ we obtain the well known Liouville fractional integral

$$A^\alpha f = \frac{1}{\Gamma(\alpha)} \int\limits_0^\infty \frac{f(x-t)}{t^{1-\alpha}} dt = \frac{1}{\Gamma(\alpha)} \int\limits_{-\infty}^{x} \frac{f(y)dy}{(x-y)^{1-\alpha}}.$$

By putting $\nu = \sum\limits_{j=0}^{\ell} \binom{\ell}{j} (-1)^j \delta_j$, $Re\,\alpha > -\ell$, with the unit Dirac masses at the points $j = 0, 1, \cdots, \ell$, one can see that $A^\alpha f$ is just the Marchaud fractional derivative of f of the order $-\alpha$ (up to a constant factor, cf. Samko, Kilbas and Marichev 1993, Rubin 1994). If ν in (0.1) is even, then a formal application of the Fourier transform leads to $|\xi|^{-\alpha} \hat{f}(\xi)$ which is the Fourier transform of the Riesz potential (Samko, Kilbas and Marichev 1993). By choosing different ν, this list of examples can be continued. One of our goals here is to show that all basic operators in fractional calculus, including derivatives of an integral order, can be represented in the form (0.1). We also give an answer to the following question: what classes of measures (or distributions) generate concrete types of fractional integrals and derivatives. The reader will see that these classes are fairly wide, and therefore new representations are very flexible. In particular, the measure ν may be absolutely continuous with the density supported by an arbitrarily small interval. Such localization seems to be useful in numerical calculations. Of course, in the case $Re\,\alpha \le 0$ the measure ν must also enjoy some cancellation properties or, in other words, ν should be a "wavelet measure". That was the reason why representations of the form (0.1) were called *wavelet type representations.* Consideration below will be carried out in the framework of L^p-spaces, $1 \le p \le \infty$ on the real line. In this connection the integral (0.1) will be usually interpreted as the limit

$$A^\alpha f = \lim_{\substack{\varepsilon \to 0 \\ \rho \to \infty}} \frac{1}{c_\nu(\alpha)} \int\limits_\varepsilon^\rho \frac{f * \nu_t}{t^{1-\alpha}} dt$$

in the L^p-norm and in the "almost everywhere"-sense.

We begin our investigation by examining the distribution

$$\int\limits_0^\infty \frac{\nu_t}{t^{1-\alpha}} dt, \quad \alpha \in \mathrm{C}$$

(see Section 1). This will allow the reader to imagine the whole picture. In Section 2 the results of Section 1 related to the case of $Re\,\alpha \leq 0$ will be presented in the context of L^p-spaces. We shall specify the classes of distributions ν (see Sections 2.2, 2.3) and give a representation to fractional derivatives, derivatives of an integral order and inverses of potentials (of the Riesz and the Feller type). The case $Re\,\alpha > 0$, when (0.1) represents various types of fractional integrals, is considered in 3. In Section 4 we give some examples.

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Notation and Preliminaries

As usually Z, N, R and C denote the set of all integers, positive integers, real numbers and complex numbers respectively; [a] is the integer part of $a \in \mathrm{R}$.

The notation $C = C(\mathrm{R}),\ L^p = L^p(\mathrm{R})$ for function spaces is standard. For the sake of convenience we sometimes write L^∞ instead of $C_0 = \{f \in C : \lim\limits_{|x|\to\infty} f(x) = 0\}$.

" Δ" stands for the Laplacian; $(f(x))_\pm = \max\{\pm f(x),\ 0\}$. $S = S(\mathrm{R})$ is the Schwartz space of test functions, S' is its dual. We denote by $\Phi = \Phi(\mathrm{R})$ the Lizorkin space of test functions. This space consists of Schwartz functions which are orthogonal to all polynomials; $\Phi' = \Phi'(\mathrm{R})$ denotes the dual of Φ. The space Φ is dense in L^p for $1 < p < \infty$ (concerning the spaces Φ, Φ' see Lizorkin 1969, Samko, Kilbas and Marichev 1993).

For $\omega \in \Phi$ and $f \in \Phi'$ the expression (f, ω) means

$$(f,\omega) = \int\limits_{-\infty}^{\infty} f(x)\overline{\omega(x)}dx, \tag{0.3}$$

where the integral can be interpreted as the value of the distribution f at the test function $\omega(x)$. We shall write "$f(x) = g(x)$ in the Φ'-sense" if $(f,\omega) = (g,\omega)$ for all $\omega \in \Phi$.

For $g \in \Phi'$, the distributions g^- and g_t are defined by

$$(g^-,\omega) = \Big(g(x),\ \omega(-x)\Big), \quad (g_t,\omega) = (g(x),\ \omega(tx)), \quad \omega \in \Phi,\ t \in \mathrm{R}. \tag{0.4}$$

In the following $\mathcal{M}$ denotes the set of all complex-valued finite Borel measures μ on the real line R. For $\mu \in \mathcal{M}$ the values $\mu(\{\pm\infty\})$ are assumed to be zero.

For the sake of convenience we use the same notation for an absolutely continuous measure μ and for its density.

(0.5)
$$\hat{\mu}(\xi) = (F\mu)(\xi) = \int_{-\infty}^{\infty} e^{ix\xi} d\mu(\xi)$$

is the Fourier transform of $\mu \in \mathcal{M} : \mu^{\vee}(\xi) = (\mathcal{F}^{-1}\mu)(\xi)$.

(0.6)
$$(H\varphi)(x) = p.v.\frac{1}{\pi i} \int_{-\infty}^{\infty} \frac{\varphi(y)}{x-y} dy$$

is the Hilbert transform of the function φ. In this notation $(H\varphi)^{\wedge}(\xi) = \hat{\varphi}(\xi) sgn\xi$ (see Neri 1971).

For $\varphi \in L^p,\ 1 \leq p \leq \infty,\ t > 0$, we define the convolution with a dilated measure $\mu \in \mathcal{M}$ by

(0.7)
$$(\varphi * \mu_t)(x) = \int_{-\infty}^{\infty} \varphi(x - ty) d\mu(y).$$

Obviously $\|\varphi * \mu_t\|_p \leq |\mu|(\mathrm{R}) \|\varphi\|_p$ where $|\mu|(\mathrm{R})$ is the total variation of $|\mu|$.

The letter c is used for constants which may assume different values at distinct occurrences. We use the symbols "." and "$\simeq$" instead of "$\leq$" and "$=$" respectively, if the relations under consideration hold up to a non-essential constant factor . Λ denotes the end of a proof.

The following definitions and auxiliary assertions will be used in the paper.

Definitions 0.1. For $\omega \in \Phi,\ \alpha \in \mathrm{C}$, *the fractional integrals* $I^{\alpha}_{\pm}\omega$ and *the fractional derivatives* $\mathcal{D}^{\alpha}_{\pm}\omega$ are defined by:

(0.8)
$$(I^{\alpha}_{\pm}\omega)(x) = [(\mp i\xi)^{-\alpha}\hat{\omega}(\xi)]^{\vee}(x),\ Re\,\alpha \geq 0,$$

(0.9)
$$(\mathcal{D}^{\alpha}_{\pm}\omega)(x) = [(\mp i\xi)^{\alpha}\hat{\omega}(\xi)]^{\vee}(x),\ Re\,\alpha \geq 0,$$

where
$$(\mp ix)^{-\alpha} = e^{\alpha \log|x| \mp (\alpha\pi i/2) sgn x}.$$

Clearly $I^{\alpha}_{\pm}\omega = \mathcal{D}^{-\alpha}_{\pm}\omega, \ \alpha \in \mathrm{C}$. For $Re\,\alpha > 0$ the operators (0.8) and (0.9) have the following classical representation

$$(0.10) \qquad (I^{\alpha}_{\pm}\omega) = \frac{1}{\Gamma(\alpha)} \int_{-\infty}^{\infty} t_{\pm}^{\alpha-1}\omega(x-t)dt = \frac{1}{\Gamma(\alpha)} \int_{0}^{\infty} t^{\alpha-1}\omega(x \mp t)dt,$$

$$(0.11) \qquad (\mathcal{D}^{\alpha}_{\pm}\omega)(x) = \frac{(\pm 1)^n}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_{0}^{\infty} t^{n-\alpha-1}\omega(x \mp t)dt, \ n = [Re\,\alpha] + 1,$$

(see Samko, Kilbas and Marichev 1993).

Definition 0.2. For $\omega \in \Phi, \ \alpha \in \mathrm{C}$ we define the *Riesz potential* $I^{\alpha}\omega$, *the conjugate Riesz potential* $I^{\alpha}_{s}\omega$, *the Riesz fractional derivative* $\mathcal{D}^{\alpha}\omega$ *and the conjugate Riesz fractional derivative* $\mathcal{D}^{\alpha}_{s}\omega$ by

$$(0.12) \qquad \begin{aligned} &(I^{\alpha}\omega)(x) = (|\xi|^{-\alpha}\hat{\omega}(\xi))^{\vee}(x), \ (I^{\alpha}_{s}\omega)(x) = (|\xi|^{-\alpha} sgn\xi\hat{\omega}(\xi))^{\vee}(x), \ Re\,\alpha \geq 0, \\ &(\mathcal{D}^{\alpha}\omega)(x) = (|\xi|^{\alpha}\hat{\omega}(\xi))^{\vee}(x), \ (\mathcal{D}^{\alpha}_{s}\omega)(x) = (|\xi|^{\alpha} sgn\xi\hat{\omega}(\xi))^{\vee}(x), \ Re\,\alpha \geq 0. \end{aligned}$$

Definition 0.3. For $f \in \Phi'$ *and* $\alpha \in \mathrm{C}$, *the* Φ'*-distributions* $I^{\alpha}_{\pm}f, \ \mathcal{D}^{\alpha}_{\pm}f, \ I^{\alpha}f, \mathcal{D}^{\alpha}f, \ I^{\alpha}_{s}f, \mathcal{D}^{\alpha}_{s}f$ are defined by duality as follows:

$$(0.13) \qquad (I^{\alpha}_{\pm}f, \omega) = (f, \overline{I^{\alpha}_{\mp}\bar{\omega}}), \quad (\mathcal{D}^{\alpha}_{\pm}f, \ \omega) = (f, \overline{\mathcal{D}^{\alpha}_{\mp}\bar{\omega}}), \quad (I^{\alpha}f, \omega) = (f, \overline{I^{\alpha}\bar{\omega}}),$$

$$(0.14) \qquad (\mathcal{D}^{\alpha}f, \omega) = (f, \overline{\mathcal{D}^{\alpha}\bar{\omega}}), \quad (I^{\alpha}_{s}f, \omega) = (f, \overline{I^{\alpha}_{s}\bar{\omega}}), \quad (\mathcal{D}^{\alpha}_{s}f, \omega) = (f, \overline{\mathcal{D}^{\alpha}_{s}\bar{\omega}}).$$

We will also consider fractional integrals of finite Borel measures $\mu \in \mathcal{M}$ defined by

$$(0.15) \qquad (I^{\alpha}_{\pm}\mu)(x) = \frac{1}{\Gamma(\alpha)} \int_{-\infty}^{\infty} (x-y)^{\alpha-1}_{\pm} d\mu(y), \quad Re\,\alpha > 0.$$

Lemma 0.4 (Rubin 1994)**.** *The integrals* $I^{\alpha}_{\pm}\mu$ *are absolutely convergent for almost all* $x \in \mathrm{R}$ *if and only if*

$$(0.16) \qquad \int_{|y|>1} y_{\mp}^{Re\,\alpha - 1} d|\mu|(y) < \infty$$

respectively.

Lemma 0.5 (Nogin and Rubin 1987). *If the functions* $f(x) \in L^r(\mathrm{R})$, $1 \le r < \infty$, *and* $g(x) \in L^p(\mathrm{R})$, $1 \le p < \infty$, *coincide in the* Φ'*-sense, then they coincide almost everywhere.*

Lemma 0.6. *Let* $\mu \in \mathcal{M}$, *and* $g(x,z)$ *be an analytic function of* $z \in G \subset \mathrm{C}$ *for* μ*-almost all* $x \in \Omega \subset \mathrm{R}$. *If there is a function* $g_0(x) \in L^1(\Omega,\mu)$ *such that* $|g(x,z)| \le g_0(x)$ *for* μ*-almost all* $x \in \Omega$ *and all* $z \in G$, *then the integral*

$$\int_{\Omega} g(x,z)d\mu(x)$$

represents an analytic function in G.

The proof of the last statement is similar to that in Gakhov 1977, p. 17.

1. Basic relations involving Φ'-distributions

Given a Φ'-distribution ν and a complex number α, consider the formal integral

$$A^{\alpha,\nu} = \int_0^\infty \frac{\nu_t}{t^{1-\alpha}} dt. \tag{1.1}$$

Our goal here is to give sense to this integral and to represent (1.1) as a linear combination of kernels arising in fractional calculus.

For $\alpha \in \mathrm{C}$, consider the following kernels (see Gelfand and Shilov 1964, p. 48):

$$h_+^\alpha(x) = \frac{x_+^{\alpha-1}}{\Gamma(\alpha)}, \quad h_-^\alpha(x) = \frac{x_-^{\alpha-1}}{\Gamma(\alpha)}, \tag{1.2}$$

$$h^\alpha(x) = \begin{cases} \dfrac{2^{-\alpha}\Gamma((1-\alpha)/2)}{\sqrt{\pi}\,\Gamma(\alpha/2)}|x|^{\alpha-1}, & \alpha \ne 1,3,5,\cdots, \\ \dfrac{|x|^{\alpha-1}\log|x|}{(-1)^{(\alpha-1)/2}\,2^{\alpha-1}\,\sqrt{\pi}\,\Gamma(\alpha/2)\Gamma((\alpha+1)/2)}, & \alpha = 1,3,5,\cdots; \end{cases} \tag{1.3}$$

$$h_s^\alpha(x) = \begin{cases} \dfrac{2^{-\alpha}\Gamma(1-\alpha/2)}{i\sqrt{\pi}\,\Gamma((\alpha+1)/2)}|x|^{\alpha-1} sgn\ x, & \alpha \ne 2,4,6,\cdots, \\ \dfrac{|x|^{\alpha-1}\log|x| sgn\ x}{i\sqrt{\pi}\,2^{\alpha-1}(-1)^{\alpha/2}\Gamma(\alpha+1)/2)\Gamma(\alpha/2)}, & \alpha = 2,4,6,\cdots. \end{cases} \tag{1.4}$$

Functions (1.2)-(1.4) agree with Φ'-distributions for which we use the same notation. These distributions have the following Fourier transforms in the Φ'-sense (see Samko, Kilbas and Marichev 1993, p. 147, Gelfand and Shilov 1964, p. 170):

(1.5) $\hat{h}^\alpha_\pm(\xi) = (\mp i\xi)^{-\alpha} = |\xi|^{-\alpha}\exp\Big(\pm\frac{\alpha\pi i}{2} sgn\ \xi\Big),\ \ \hat{h}^\alpha(\xi) = |\xi|^{-\alpha},\ \ \hat{h}^\alpha_s(\xi) = sgn\ \xi|\xi|^{-\alpha}.$

Clearly, $I^\alpha_\pm f = h^\alpha_\pm * f,\ I^\alpha f = h^\alpha * f,\ I^\alpha_s f = h^\alpha_s * f$ for all $\alpha \in$ C and $f \in \Phi'$. The equalities (1.5) lead to the following relations between fractional integrals:

$$I^\alpha_\pm f = \cos(\alpha\pi) I^\alpha_\mp f \pm i\sin(\alpha\pi) H I^\alpha_\mp f, \quad I^\alpha_s f = H I^\alpha f, \tag{1.6}$$

$$I^\alpha f = H I^\alpha_s f, \quad I^\alpha_\pm f = \cos(\alpha\pi/2) I^\alpha f \pm i\ \sin(\alpha\pi/2) I^\alpha_s f, \tag{1.7}$$

$$I^\alpha f = \frac{1}{2\cos(\alpha\pi/2)}(I^\alpha_+ f + I^\alpha_- f),\ \alpha \ne 2k+1, \quad I^{2k+1}_\pm f = \pm i(-1)^k I^{2k+1}_s f, \tag{1.8}$$

$$I^\alpha_s f = \frac{1}{2i\sin(\alpha\pi/2)}(I^\alpha_+ f - I^\alpha_- f),\ \alpha \ne 2k, \quad I^{2k}_\pm f = (-1)^k I^{2k} f. \tag{1.9}$$

These relations hold for all $\alpha \in$ C and $f \in \Phi'$.

Denote

$$A^{\alpha,\nu}_{\varepsilon,\rho} = \int_\varepsilon^\rho \frac{\nu_t}{t^{1-\alpha}} dt, \quad 0 < \varepsilon < \rho < \infty.$$

Lemma 1.1. *Let $\alpha \in$ C, ν be a Φ'-distribution such that $\hat{\nu}(\xi) \in L_{\mathrm{loc}}(\mathrm{R}\setminus\{0\})$. Denote $\hat{\nu}_\pm(\xi) = \dfrac{\hat{\nu}(\xi) \pm \hat{\nu}(-\xi)}{2}$ and let the integrals*

$$a_\pm = \int_0^\infty \frac{\hat{\nu}_\pm(\eta)}{\eta^{1-\alpha}} d\eta \tag{1.10}$$

exist as the improper ones. Then $A^{\alpha,\nu}_{\varepsilon,\rho} \to a_+ h^\alpha + a_- h^\alpha_s$ as $\varepsilon \to 0,\ \rho \to \infty$ in the Φ'-sense. If $\alpha \notin$ Z, then this limit is also equal to $c_+ h^\alpha_+ + c_- h^\alpha_-$, where $c_\pm = \frac{1}{2}\Big[a_+/\cos(\frac{\alpha\pi}{2}) \pm a_-/\sin(\frac{\alpha\pi}{2})\Big]$.

P r o o f. For $\omega \in \Phi$, we have

$$\Big(A^{\alpha,\nu}_{\varepsilon,\rho}, \omega\Big) = \frac{1}{2\pi}\int_\varepsilon^\rho \frac{(\hat{\nu}(t\xi), \hat{\omega}(\xi))}{t^{1-\alpha}} dt = \frac{1}{2\pi}\int_\varepsilon^\rho \frac{(\hat{\nu}_+(t\xi), \hat{\omega}(\xi))}{t^{1-\alpha}} dt + \frac{1}{2\pi}\int_\varepsilon^\rho \frac{(\hat{\nu}_-(t\xi), \hat{\omega}(\xi))}{t^{1-\alpha}} dt =$$

$$= \frac{1}{2\pi}\Big(|\xi|^{-\alpha}\int_{\varepsilon|\xi|}^{\rho|\xi|} \frac{\hat{\nu}_+(\eta)}{\eta^{1-\alpha}} d\eta, \hat{\omega}(\xi)\Big) + \frac{1}{2\pi}\Big(|\xi|^{-\alpha} sgn\ \xi \int_{\varepsilon|\xi|}^{\rho|\xi|} \frac{\hat{\nu}_-(\eta)}{\eta^{1-\alpha}} d\eta, \hat{\omega}(\xi)\Big).$$

Since the integrals (1.10) are finite, we obtain :

$$\lim_{\substack{\varepsilon\to 0\\ \rho\to\infty}}^{(\Phi')} \left(A^{\alpha,\nu}_{\varepsilon,\rho},\omega\right) = \frac{a_+}{2\pi}(\hat{h}^\alpha(\xi),\hat{\omega}(\xi)) + \frac{a_-}{2\pi}(\hat{h}^\alpha_s(\xi),\hat{\omega}(\xi)) = a_+(h^\alpha,\omega) + a_-(h^\alpha_S,\omega).$$

The second statement is an obvious consequence of the following relations (see the first relation in (1.5)): $\hat{h}^\alpha_+ + \hat{h}^\alpha_- = 2\hat{h}^\alpha \cos\frac{\alpha\pi}{2}$, $\hat{h}^\alpha_+ - \hat{h}^\alpha_- = 2\hat{h}^\alpha_s \sin\frac{\alpha\pi}{2}$. Λ

Remark 1.2. If α is an integer, then one can also write

$$\lim_{\substack{\varepsilon\to 0\\ \rho\to\infty}}^{(\Phi')} A^{\alpha,\nu}_{\varepsilon,\rho} = (-1)^k(a_+ + a_- H)h^{2k}_\pm, \quad \alpha = 2k, \; k\in Z,$$

$$\lim_{\substack{\varepsilon\to 0\\ \rho\to\infty}}^{(\Phi')} A^{\alpha,\nu}_{\varepsilon,\rho} = i(-1)^{k+1}(a_+H + a_-)h^{2k+1}_+ = i(-1)^k(a_+H + a_-)h^{2k+1}_-, \;\alpha = 2k+1, \;\; k\in Z,$$

where H stands for the Hilbert transform (0.6).

These equalities can be easily derived from the following relations in the Fourier terms:

$$\hat{h}^{2k}(\xi) = (-1)^k\hat{h}^{2k}_\pm(\xi), \quad \hat{h}^{2k}_s(\xi) = (-1)^k\hat{h}^{2k}_\pm(\xi) sgn\;\xi,$$

$$\hat{h}^{2k+1}(\xi) = i(-1)^{k+1}\hat{h}^{2k+1}_+(\xi) sgn\;\xi = i(-1)^k\hat{h}^{2k+1}_-(\xi) sgn\;\xi,$$

$$\hat{h}^{2k+1}_s(\xi) = i(-1)^{k+1}\hat{h}^{2k+1}_+(\xi) = i(-1)^k\hat{h}^{2k+1}_-(\xi).$$

Lemma 1.1 shows that the integral $\int_0^\infty(\varphi*\nu_t)(x)\,dt/t^{1-\alpha}$ may be used for representation of the following operators:

(a) Riesz potentials and their inverses ($a_+ = 1, a_- = 0$);

(b) conjugate Riesz potentials and their inverses ($a_+ = 0, a_- = 1$);

(c) left-sided fractional integrals and derivatives ($c_+ = 1, c_- = 0$);

(d) right-sided fractional integrals and derivatives ($c_+ = 0, c_- = 1$);

(e) integrals and derivatives of an integral order (the related formulae for coefficients are obvious from (1.8), (1.9)).

In a similar way one can represent compositions of mentioned operators with a Hilbert transform and linear combinations (with constant coefficients) of such operators.

2. Wavelet Type Representation of Fractional Derivatives (L^p-theory)

In this section we obtain natural L^p-analogues of Lemma 1.1 and describe some classes of measures ν for which these analogues hold. The main results are presented in Theorems 2.8, 2.11.

For convenience of the reader we first sketch the basic steps of our program without going into details. Given a suitable measure ν and a test function $\omega \in \Phi$, we find a kernel $k(x)$ for which

$$\int_{\varepsilon}^{\infty} \frac{(\nu_t, \omega)}{t^{1+\alpha}} dt = (k_\varepsilon, \psi), \quad Re\, \alpha > 0,$$

where $k_\varepsilon(x) = \dfrac{1}{\varepsilon} k(\dfrac{x}{\varepsilon})$ and $\psi = \mathcal{D}_-^\alpha \omega$ is the right-sided Liouville fractional derivative. We will see that $k(x) = (1/x) \int_0^x \mu(y) dy$ with $\mu = I_+^\alpha \nu \in L^1$.

Put $k = k^+ + k^-$, where $k^\pm(x) = (1/2x) \int_0^x [\mu(y) \pm \mu(-y)] dy$. The kernel k^+ turns out to be summable and $k^-(x)$ has a "bad" behaviour at infinity. We correct $k^-(x)$ by putting $k^-(x) = h(x) - \lambda \pi i q(x)$ with a suitable coefficient λ and the conjugate Poisson kernel $q(x) = \dfrac{1}{\pi i} \dfrac{x}{1+x^2}$. A new function $h(x)$ will then be summable and $\int_{-\infty}^{\infty} h(x) dx = 0$. As a result we get

$$\int_{\varepsilon}^{\infty} \frac{(\nu_t, \omega)}{t^{1+\alpha}} dt = A(\varepsilon), \quad A(\varepsilon) = (k_\varepsilon^+, \psi) + (h_\varepsilon, \psi) - \lambda \pi i (q_\varepsilon, \psi).$$

This formula enables us to obtain relevant relations for L^p-functions. Then one can apply usual tools of approximation to the identity and the well-known fact that the Hilbert transform can be approximated by conjugate Poisson integrals (Neri 1971).

If ν is a distribution, certain modifications of this scheme should be used (see Section 2.4).

2.1. Auxiliary relations.

Lemma 2.1. *Let $Re\,\alpha > 0$, ν be a finite Borel measure such that*

$$(2.1)\qquad \int_{-\infty}^{\infty} x^j d\nu(x) = 0\ \forall j = 0, 1, \cdots, m; \quad m = \begin{cases} \alpha - 1 & \text{if } \alpha \in \mathrm{N}, \\ [Re\,\alpha] & \text{otherwise.} \end{cases}$$

(i) *If $Re\,\alpha \notin \mathrm{N}$ or α is an integer, then $I^\alpha_\pm \nu \in L^1$ provided*

$$(2.2)\qquad \int_{|x|>1} |x|^{Re\,\alpha} d|\nu|(x) < \infty.$$

(ii) *If $Re\,\alpha \in \mathrm{N}$, $Im\,\alpha \neq 0$, then $I^\alpha_+ \nu \in L^1$ provided*

$$(2.3)\qquad \int_{-\infty}^{-1} |x|^{Re\,\alpha} d|\nu|(x) < \infty, \quad \int_{1}^{\infty} x^{Re\,\alpha} \log x \; d|\nu|(x) < \infty,$$

and $I^\alpha_- \nu \in L^1$ provided

$$(2.3')\qquad \int_{-\infty}^{-1} |x|^{Re\,\alpha} \log|x| \; d|\nu|(x) < \infty, \quad \int_{1}^{\infty} x^{Re\,\alpha} d|\nu|(x) < \infty.$$

P r o o f. Consider $I^\alpha_+\nu$. We show separately that $I^\alpha_+\nu \in L^1(-\infty, 1)$ and $I^\alpha_+\nu \in L^1(1, \infty)$. Put $\alpha' = Re\,\alpha$. The first relation is obvious:

$$\int_{-\infty}^{1} |(I^\alpha_+\nu)(x)|dx \;.\; \int_{-\infty}^{1} d|\nu|(y) \int_{y}^{1} (x-y)^{\alpha'-1} dx = \frac{1}{\alpha'} \int_{-\infty}^{1} (1-y)^{\alpha'} d|\nu|(y) < \infty.$$

In order to prove the second relation one can use the Taylor expansion:

$$\frac{(x-y)^{\alpha-1}}{\Gamma(\alpha)} = \sum_{j=0}^{m} \frac{(-y)^j}{j!} \frac{x^{\alpha-1-j}}{\Gamma(\alpha-j)} + \frac{(-1)^{m+1}}{m!\Gamma(\alpha-m+1)} \int_{0}^{y} (y-t)^m (x-t)^{\alpha-m-2} dt$$

with m defined in (2.1), (in the case $\alpha \in \mathrm{N}$ the integral in the right-hand side disappears).

In accordance with (2.1) we have:

$$\int_1^\infty |(I_+^\alpha \nu)(x)|dx \leq \sum_{j=0}^m \frac{1}{j!|\Gamma(\alpha - j)|} \int_1^\infty \Big| x^{\alpha-1-j} \int_x^\infty y^j d\nu(y) \Big| dx +$$

$$+ \int_1^\infty \Big| \frac{1}{m!\Gamma(\alpha - m - 1)} \int_{-\infty}^x d\nu(y) \int_0^y (y-t)^m (x-t)^{\alpha-m-2} dt \Big| dx \, .$$

$$. \sum_{j=0}^m \int_1^\infty x^{\alpha'-1-j} dx \int_x^\infty y^j d|\nu|(y) + \int_1^\infty dx \int_{-\infty}^0 d|\nu|(y) \int_0^{|y|} (|y|-t)^m (x+t)^{\alpha'-m-2} dt +$$

$$+ \int_1^\infty dx \int_0^x d|\nu|(y) \int_0^y (y-t)^m (x-t)^{\alpha'-m-2} dt.$$

By changing the order of integration one can easily get

$$\int_1^\infty |(I_+^\alpha \nu)(x)|dx \, . \sum_{j=0}^m \int_1^\infty y^j (y^{\alpha'-j} - 1) d|\nu|(y) + \int_{-\infty}^0 u(y) d|\nu|(y) + \int_0^1 v(y) d|\nu|(y), \tag{2.4}$$

where

$$u(y) = \int_0^{|y|} (|y|-t)^m (1+t)^{\alpha'-m-1} dt, \;\; v(y) = \int_0^y (y-t)^m (1-t)^{\alpha'-m-1} dt.$$

The first term in (2.4) is finite. This completes the proof for the case $\alpha \in \mathrm{N}$. If $Re\, \alpha \notin \mathrm{N}$, then

$$u(y) \leq |y|^m \int_0^{|y|} t^{\alpha'-m-1} dt \simeq \; |y|^{\alpha'}.$$

If $Re\, \alpha \in \mathrm{N}$, $Im\, \alpha \neq 0$ (in this case $\alpha' - m = 0$), then

$$u(y) = \int_0^{|y|} (|y|-t)^m \frac{dt}{t+1} = |y|^m \int_0^1 (1-\tau)^m \frac{d\tau}{\tau + \frac{1}{|y|}} \leq |y|^m \int_0^1 \frac{d\tau}{\tau + \frac{1}{|y|}} = |y|^m \log(1+|y|).$$

Thus by (2.2), (2.3) the second term in (2.4) is finite. The finiteness of the third one is implied by the simple estimate

$$v(y) \leq \int_0^y (1-t)^{\alpha'-1} < \infty \quad \text{for } y < 1.$$

For $I_-^\alpha \nu$ the result can be deduced from that for $I_+^\alpha \nu$ by changing variables. Λ

Lemma 2.2. *Let* $Re\,\alpha > 0$ *and* ν *satisfy the conditions of Lemma 2.1. If* $\omega \in \Phi$ *and* $\psi = \overline{\mathcal{D}_-^\alpha \bar{\omega}}$, *then*

$$\int\limits_{\varepsilon}^{\infty} \frac{(\nu_t, \omega)}{t^{1+\alpha}} dt = (k_\varepsilon^+, \psi) + (k_\varepsilon^-, \psi), \quad 0 < \varepsilon < \infty, \tag{2.5}$$

$$k^+(x) = \frac{1}{2x}[(I_+^{1+\alpha}\nu)(x) - (I_+^{1+\alpha}\nu)(-x)], \tag{2.6}$$

$$k^-(x) = \frac{1}{2x}[(I_+^{1+\alpha}\nu)(x) + (I_+^{1+\alpha}\nu)(-x)] - \frac{\lambda}{x}, \quad \lambda = \frac{1}{\Gamma(1+\alpha)} \int\limits_{-\infty}^{0} |x|^\alpha d\nu(x). \tag{2.7}$$

P r o o f. By Lemma 2.1, $\mu = I_+^\alpha \nu \in L^1$ and therefore $\nu = \mathcal{D}_+^\alpha \mu$ in the Φ'-sense. It follows that

$$\int\limits_{\varepsilon}^{\infty} \frac{(\nu_t, \omega)}{t^{1+\alpha}} dt = \int\limits_{\varepsilon}^{\infty} \frac{(\mu_t, \psi)}{t} dt = \int\limits_{-\infty}^{\infty} \overline{\psi(x)} dx \begin{Bmatrix} \frac{1}{x} \int\limits_{0}^{x/\varepsilon} \mu(y)dy, \ x > 0 \\ \frac{1}{x} \int\limits_{x/\varepsilon}^{0} \mu(y)dy, \ x < 0 \end{Bmatrix} = (k_\varepsilon, \psi),$$

where $k(x) = \dfrac{1}{x}\displaystyle\int\limits_0^x \mu(y)dy$. Put $\mu = \mu^+ + \mu^-$, $\mu^\pm(x) = [\mu(x) \pm \mu(-x)]/2$ and define

$$k^\pm(x) = \frac{1}{x}\int\limits_0^x \mu^\pm(y)dy = \frac{1}{2x}\int\limits_0^x [(I_+^\alpha\nu)(y) \pm (I_+^\alpha\nu)(-y)]dy. \tag{2.8}$$

Note that $k^+(x)$ is an even function and $k^-(x)$ is an odd one.

By (2.2), the fractional integral $(I_+^{1+Re\,\alpha}|\nu|)(x)$ exists for almost all x (cf. Lemma 0.4), and therefore one can change the order of integration in (2.8). For $x > 0$ we have

$$k^+(x) = \frac{1}{2\Gamma(\alpha)x} \int\limits_0^x dy \Big[\int\limits_{-\infty}^{y} (y-t)^{\alpha-1} d\nu(t) + \int\limits_{-\infty}^{-y} (-y-t)^{\alpha-1} d\nu(t) \Big] =$$

$$= \frac{1}{2x\Gamma(1+\alpha)} \Big[\int\limits_{-\infty}^{0} [(x-t)^\alpha - |t|^\alpha] d\nu(t) + \int\limits_0^x (x-t)^\alpha d\nu(t) + \int\limits_{-\infty}^{-x} [|t|^\alpha - (-x-t)^\alpha] d\nu(t) +$$

$$+ \int\limits_{-x}^{0} |t|^\alpha d\nu(t) \Big] = \frac{1}{2x\Gamma(\alpha+1)} \Big[\int\limits_{-\infty}^{x} (x-t)^\alpha d\nu(t) - \int\limits_{-\infty}^{-x} (-x-t)^\alpha d\nu(t) \Big].$$

This coincides with (2.6). The proof of (2.7) is similar. Λ

Lemma 2.3. *Let* $Re\,\alpha > 0$, $\nu \in \mathcal{M}$. *If*

$$(2.9)\qquad \int_{-\infty}^{\infty} x^j d\nu(x) = 0 \ \forall j = 0, 1, \cdots, [Re\,\alpha], \quad \int_{|x|>1} |x|^\beta d|\nu|(x) < \infty \text{ for some } \beta > Re\,\alpha,$$

then for sufficiently small $\varepsilon_1, \varepsilon_2 > 0$ *the following estimates are valid:*

$$(2.10)\qquad \left|\frac{1}{x}\left(I_+^{1+\alpha}\nu\right)(x)\right| \lesssim |x|^{-\varepsilon_1 - 1}, \quad |x| > 1 \quad \text{and} \quad |k^\pm(x)| \lesssim |x|^{\varepsilon_2 - 1}, \quad |x| < 1.$$

The proof of this lemma is technical and given in Appendix 1. Now from (2.7) and the first inequality in (2.10) we see that k^- is not summable because of the term $\dfrac{\lambda}{x}$. We make a correction of k^- by considering another kernel:

$$(2.11)\qquad h(x) = k^-(x) + \lambda\pi i q(x), \quad q(x) = \frac{1}{\pi i}\frac{x}{1+x^2}$$

($q(x)$ is the conjugate Poisson kernel).

Lemma 2.4. *Let* α *and* ν *be the same as in Lemma 2.3. Then*

$$(2.12)\qquad h \in L^1, \quad \int_{-\infty}^{\infty} h(x)dx = 0, \quad |h(x)| \lesssim \begin{cases} |x|^{\varepsilon_1 - 1}, & |x| < 1 \\ |x|^{-\varepsilon_1 - 1}, & |x| > 1, \end{cases} \quad \varepsilon_1, \varepsilon_2 > 0.$$

P r o o f. The behaviour of $h(x)$ for $|x| < 1$ is obvious from (2.10), (2.11) by taking into account that

$$(2.13)\qquad -\frac{\lambda}{x} + \lambda\pi i q(x) = -\frac{\lambda}{x(1+x^2)} = 0(|x|^{-3}).$$

The integral of h is zero because h is odd. Λ

Lemma 2.5. *Let* $Re\,\alpha > 0$ *and* ν *satisfy the conditions of Lemma 2.3. Then for* $\omega \in \Phi$, $\psi = \overline{\mathcal{D}_-^\alpha \bar{\omega}}$, $g = k^+ + h - \lambda\pi i q$, *the following relation holds:*

$$(2.14)\qquad \int_\varepsilon^\infty \frac{(\nu_t, \omega)}{t^{1+\alpha}} dt = (g_\varepsilon, \psi).$$

It follows that

$$(2.15)\qquad \lim_{\varepsilon\to 0}\int_\varepsilon^\infty \frac{(\nu_t, \omega)}{t^{1+\alpha}} dt = \gamma_+\Big(\frac{x_+^{-\alpha-1}}{\Gamma(-\alpha)}, \omega\Big) + \gamma_-\Big(\frac{x_+^{-\alpha-1}}{\Gamma(-\alpha)}, H\omega\Big),$$

where H stands for the Hilbert transform,

$$\gamma_+ = \int_{-\infty}^{\infty} k^+(x)dx = \begin{cases} \Gamma(-\alpha)\Big[\int_0^{\infty} x^\alpha d\nu(x) + \cos\alpha\pi \int_{-\infty}^{0} |x|^\alpha d\nu(x)\Big], & \alpha \notin \mathrm{N}, \\ \dfrac{(-1)^\ell}{\ell!} \int_{-\infty}^{\infty} x^\ell \log\dfrac{1}{|x|} d\nu(x), & \alpha = \ell \in \mathrm{N}, \end{cases} \tag{2.16}$$

$$\gamma_- = -\lambda\ \pi i = -\frac{\pi i}{\Gamma(1+\alpha)} \int_{-\infty}^{0} |x|^\alpha d\nu(x). \tag{2.17}$$

P r o o f. The relation (2.14) follows from (2.5), (2.11). By taking into account Lemma 2.3, 2.4, we obtain

$$\lim_{\varepsilon\to 0}(k_\varepsilon^+, \psi) = \gamma_+\overline{\psi(0)} = \gamma_+(\delta, \psi) = \gamma_+(\mathcal{D}_+^\alpha \delta, \omega) = \gamma_+\Big(\frac{x_+^{-\alpha-1}}{\Gamma(-\alpha)}, \omega\Big), \quad \lim_{\varepsilon\to 0}(h_\varepsilon, \psi) = 0,$$

$$\lim_{\varepsilon\to 0}(q_\varepsilon, \psi) = \lim_{\varepsilon\to 0}\frac{1}{2\pi}\Big(e^{-\varepsilon|\xi|} sgn\ \xi, \hat{\psi}(\xi)\Big) = \Big(\frac{x_+^{-\alpha-1}}{\Gamma(-\alpha)},\ H\omega\Big).$$

It remains to verify the second relations in (2.16). We refer the reader to Appendix 3 for these calculations. Λ

The equality (2.15) will lead us to the wavelet type representation of the operator $\gamma_+\mathcal{D}_+^\alpha + \gamma_- H\mathcal{D}_+^\alpha$ (see Section 2.2). In order to obtain similar representations for linear combinations like $c_+\mathcal{D}_-^\alpha + c_-\mathcal{D}_-^\alpha$ or $a_+\mathcal{D}^\alpha + a_-\mathcal{D}_S^\alpha$, (with Riesz derivatives) we rewrite the right-hand side of (2.14) in suitable form.

Lemma 2.6. *Let $\alpha, \nu, \lambda, \omega(x)$ and $h(x)$ be the same as in Lemma 2.5.*

(i) *If $\alpha \notin \mathrm{N}$, then*

$$\int_{\varepsilon}^{\infty} \frac{(\nu_t, \omega)}{t^{1+\alpha}} dt = (u_\varepsilon^+, \overline{\mathcal{D}_-^\alpha \bar{\omega}}) + (u_\varepsilon^-, \overline{\mathcal{D}_+^\alpha \bar{\omega}}) + (h_\varepsilon, \overline{\mathcal{D}_-^\alpha \bar{\omega}}), \tag{2.18}$$

where

$$u^+(x) = k^+(x) + \lambda\pi\mathcal{P}(x)cot(\alpha\pi), \quad \mathcal{P}(x) = \frac{1}{\pi(1+x^2)}\ (\textit{the Poisson kernel}),$$
$$u^-(x) = -\frac{\lambda\pi}{\sin(\alpha\pi)}\mathcal{P}(x). \tag{2.18'}$$

It follows that

$$\lim_{\varepsilon\to 0}\int_{\varepsilon}^{\infty}\frac{(\nu_t,\omega)}{t^{1+\alpha}}dt = c_+\Big(\frac{x_+^{-\alpha-1}}{\Gamma(-\alpha)},\omega\Big)+c_-\Big(\frac{x_-^{-\alpha-1}}{\Gamma(-\alpha)},\omega\Big), \tag{2.19}$$

where

$$c_+=\Gamma(-\alpha)\int_0^{\infty}x^{\alpha}d\nu(x),\quad c_-=\Gamma(-\alpha)\int_{-\infty}^{0}|x|^{\alpha}d\nu(x). \tag{2.20}$$

(ii) For any $Re\,\alpha>0$,

$$\int_{\varepsilon}^{\infty}\frac{(\nu_t,\omega)}{t^{1+\alpha}}dt=\Big(v_\varepsilon^+,\overline{\mathcal{D}^{\alpha}\bar\omega}\Big)+\Big(v_\varepsilon^-,\overline{\mathcal{D}_S^{\alpha}\bar\omega}\Big)+\Big(h_\varepsilon,\overline{\mathcal{D}_-^{\alpha}\bar\omega}\Big), \tag{2.21}$$

where

$$v^+=\cos\frac{\alpha\pi}{2}k^+(x)-\lambda\pi\sin\frac{\alpha\pi}{2}\mathcal{P}(x),\qquad v^-=-i\sin\frac{\alpha\pi}{2}k^+(x)-\lambda\pi\ i\cos\frac{\alpha\pi}{2}\mathcal{P}(x).$$

It follows that

$$\lim_{\varepsilon\to 0}\int_{\varepsilon}^{\infty}\frac{(\nu_t,\omega)}{t^{1+\alpha}}dt=a_+(h^{-\alpha},\omega)+a_-(h_S^{-\alpha},\omega), \tag{2.22}$$

where distributions $h^{-\alpha}$, $h_s^{-\alpha}$ *(of the Riesz type) are defined by* (1.3), (1.4) *and the coefficients* $a_\pm$ *can be calculated as follows.*

For $\alpha\notin$ N:

$$a_+=\Gamma(-\alpha)\cos\frac{\alpha\pi}{2}\int_{-\infty}^{\infty}|x|^{\alpha}d\nu(x),\quad a_-=-i\Gamma(-\alpha)\sin\frac{\alpha\pi}{2}\int_{-\infty}^{\infty}|x|^{\alpha}sgn\,x\ d\nu(x). \tag{2.23}$$

For $\alpha=\ell\in$ N:

$$a_+=\begin{cases}\dfrac{(-1)^{\ell/2}}{\ell!}\int_{-\infty}^{\infty}x^{\ell}\log\dfrac{1}{|x|}d\nu(x), & \ell=2k, k=1,2,\cdots,\\ \dfrac{\pi(-1)^{(\ell-1)/2}}{\ell!}\int_{-\infty}^{0}x^{\ell}d\nu(x), & \ell=2k+1, k=0,1,2,\cdots;\end{cases} \tag{2.24}$$

$$(2.24') \qquad a_- = \begin{cases} \dfrac{i(-1)^{\ell/2+1}\pi}{\ell!} \int\limits_{-\infty}^{0} x^\ell d\nu(x), \ \ell = 2k, k = 1, 2, \cdots, \\ \dfrac{i(-1)^{(\ell-1)/2}}{\ell!} \int\limits_{-\infty}^{\infty} x^\ell \log \dfrac{1}{|x|} d\nu(x), \ \ell = 2k+1, k = 0, 1, 2, \cdots, \end{cases}$$

P r o o f. (i) The equality (2.18) is a consequence of (2.14) and the relation:

$$(2.25) \qquad \mathcal{D}_-^\alpha \omega = H(H\mathcal{D}_-^\alpha \omega) = H(\frac{\cot(\alpha\pi)}{i}\mathcal{D}_-^\alpha \omega - \frac{\mathcal{D}_+^\alpha \omega}{i\sin(\alpha\pi)}), \quad \alpha \notin \mathrm{N},$$

which follows from (1.7).

By taking into account Lemmas 2.3, 2.4, 2.5, we obtain

$$\lim_{\varepsilon\to 0}(u_\varepsilon^+, \overline{\mathcal{D}_-^\alpha \bar\omega}) = (\gamma_+ + \lambda\pi\cot(\alpha\pi))(\frac{x_+^{-\alpha-1}}{\Gamma(-\alpha)}, \omega) \quad \text{with } \gamma_+ \text{ as in (2.16)},$$

$$\lim_{\varepsilon\to 0}(h_\varepsilon, \overline{\mathcal{D}_-^\alpha \bar\omega}) = 0, \quad \lim_{\varepsilon\to 0}(u_\varepsilon^-, \overline{\mathcal{D}_+^\alpha \bar\omega}) = -\frac{\lambda\pi}{\sin(\alpha\pi)}\Big(\frac{x_-^{-\alpha-1}}{\Gamma(-\alpha)}, \omega\Big).$$

Thus we have (2.19) with

$$c_+ = \gamma_+ + \lambda\pi\cot(\alpha\pi) = \Gamma(-\alpha)\int\limits_0^\infty x^\alpha d\nu(x), \quad c_- = -\frac{\lambda\pi}{\sin(\alpha\pi)} = \Gamma(-\alpha)\int\limits_{-\infty}^0 |x|^\alpha d\nu(x).$$

(ii) The relation (2.21) follows from (2.14) because

$$(2.26) \qquad \mathcal{D}_-^\alpha \omega = \cos\frac{\alpha\pi}{2}\mathcal{D}^\alpha\omega + i\sin\frac{\alpha\pi}{2}H\mathcal{D}^\alpha\omega \qquad \text{(see (1.6), (1.9))}.$$

As above we have

$$\lim_{\varepsilon\to 0}(v_\varepsilon^+, \overline{\mathcal{D}^\alpha \bar\omega}) = (\cos\frac{\alpha\pi}{2}\gamma_+ - \lambda\pi\sin\frac{\alpha\pi}{2})(h^{-\alpha}, \omega),$$
$$\lim_{\varepsilon\to 0}(h_\varepsilon, \overline{\mathcal{D}_-^\alpha \bar\omega}) = 0, \lim_{\varepsilon\to 0}(v_\varepsilon^-, \overline{H\mathcal{D}^\alpha \bar\omega}) = -i(\sin\frac{\alpha\pi}{2}\gamma_+ + \lambda\pi\cos\frac{\alpha\pi}{2})(h_S^{-\alpha}, \omega).$$

Thus we have (2.22) with $a_+ = \cos\frac{\alpha\pi}{2}\gamma_+ - \lambda\pi\sin\frac{\alpha\pi}{2}$, $a_- = -i\Big(\sin\frac{\alpha\pi}{2}\gamma_+ + \lambda\pi\cos\frac{\alpha\pi}{2}\Big)$. By (2.16), (2.17) these coefficients can be transformed into those in (2.23)-(2.24′). Λ

2.2. The main theorem.

Now we are ready to present the main result for the integrals

$$\int\limits_0^\infty \frac{\nu_t * f}{t^{1+\alpha}}dt, \ Re\,\alpha > 0.$$

Below we use the function $k^+(x), h(x), q(x)$ and the constants $\lambda, \gamma_\pm, c_\pm, a_\pm$ defined in the preceding section.

Lemma 2.7. *Let $f \in \Phi'$, $\alpha \in \mathrm{C}$. If one of the distributions $I^\alpha_+ f$, $I^\alpha_- f$, $I^\alpha f$, $I^\alpha_s f$ belongs to L^p, $1 < p < \infty$, then all the rest also belong to L^p and their L^p-norms are equivalent.*

This statement is obvious from (1.6)-(1.9) by taking into account the boundedness of the Hilbert transform H in L^p for $1 < p < \infty$.

Theorem 2.8. *Let $Re\,\alpha > 0$, $\nu \in \mathcal{M}$. Assume that*

$$\int\limits_{-\infty}^{\infty} x^j d\nu(x) = 0 \quad \text{for} \quad j = 0, 1, \cdots, [Re\,\alpha], \qquad \int\limits_{|x|>1} |x|^\beta d|\nu|(x) < \infty \tag{2.27}$$

for some $\beta > Re\,\alpha$. Let $f \in L^r$, $1 \le r < \infty$, and one of the derivatives $\mathcal{D}^\alpha_\pm f, D^\alpha f$, $D^\alpha_s f$ (in the Φ'-sense) belongs to L^p, $1 < p < \infty$. Then the limit

$$A^\alpha f = \lim_{\varepsilon \to 0} \int\limits_{\varepsilon}^{\infty} \frac{\nu_t * f}{t^{1+\alpha}} dt \tag{2.28}$$

exists in the L^p-norm and in the a.e. sense, and the following relations hold:

$$A^\alpha f = \gamma_+ \mathcal{D}^\alpha_+ f + \gamma_- H \mathcal{D}^\alpha_+ f = c_+ \mathcal{D}^\alpha_+ f + c_- \mathcal{D}^\alpha_- f = a_+ \mathcal{D}^\alpha f + a_- \mathcal{D}^\alpha_s f. \tag{2.29}$$

(in the second equality it is assumed $\alpha \notin \mathrm{N}$).

P r o o f. By (1.6)-(1.7) the expressions in (2.29) coincide in the Φ'-sense. By Lemmas 2.6, 0.5 they also coincide almost everywhere. Thus, it suffices to verify the first equality in (2.29). Take $\omega \in \Phi$ and denote $\omega_x(y) = \omega(x+y), \psi_x(y) = \overline{(\mathcal{D}^\alpha_- \bar{\omega}_x)}(y) = \overline{(\mathcal{D}^\alpha_- \bar{\omega})}(x+y)$. Then by (2.14),

$$\int\limits_{\varepsilon}^{\infty} \frac{(\nu_t, \omega_x)}{t^{1+\alpha}} dt = (g_\varepsilon, \psi_x), \quad g = k^+ + h - \lambda \pi i q, \quad g_\varepsilon(x) = \varepsilon^{-1} g(x/\varepsilon). \tag{2.29'}$$

This gives

$$\Big(\int\limits_{\varepsilon}^{\infty} \frac{\nu_t * f}{t^{1+\alpha}} dt,\ \omega \Big) = \Big(f(x),\ \overline{\int\limits_{\varepsilon}^{\infty} \frac{(\nu_t, \omega_x)}{t^{1+\alpha}} dt} \Big) = (f(x), \overline{(g_\varepsilon, \psi_x)}) = (g_\varepsilon * f, \overline{\mathcal{D}^\alpha_- \bar{\omega}}). \tag{2.30}$$

Given an arbitrary function $a(x) \in L^s$, $1 \le s < \infty$, the expression

$$(a_\varepsilon * f, \psi), \quad \text{where } \psi = \overline{\mathcal{D}^\alpha_- \bar{\omega}}, \tag{2.31}$$

can be transformed into $(a_\varepsilon * \mathcal{D}_+^\alpha f,\ \omega)$. Indeed, for any sequence $\{a_m(x)\} \in C_0^\infty$ converging to $a(x)$ in the L^s-norm we have

$$(a_\varepsilon * f,\ \psi) = \lim_{m\to\infty} ((a_m)_\varepsilon * f, \psi) = \lim_{m\to\infty} (f, \overline{(a_m^-)_\varepsilon * \bar{\psi}}),$$

where $a_m^-(x) = a_m(-x)$. Since a_m is a multiplier in Φ, then

$$(a_m^-)_\varepsilon * \bar{\psi} = (a_m^-)_\varepsilon * \mathcal{D}_-^\alpha \bar{\omega} = \mathcal{D}_-^\alpha[(a_m^-)_\varepsilon * \bar{\omega}] \in \Phi,$$

and by taking into account that $\mathcal{D}_+^\alpha f \in L^p$, we obtain

$$(a_\varepsilon * f,\ \psi) = \lim_{m\to\infty} (f, \overline{\mathcal{D}_-^\alpha[(a_m^-)_\varepsilon * \bar{\omega}]}) = \lim_{m\to\infty} (\mathcal{D}_+^\alpha f, \overline{(a_m^-)_\varepsilon * \bar{\omega}}) =$$
$$= \lim_{m\to\infty} ((a_m)_\varepsilon * \mathcal{D}_+^\alpha f,\ \omega) = (a_\varepsilon * \mathcal{D}_+^\alpha f,\ \omega).$$

The expression (2.30) is a linear combination of those of the form (2.31), because $k^+ \in L^1,\ h \in L^1,\ q \in L^s\ \forall s > 1$. It follows that

$$\Big(\int_\varepsilon^\infty \frac{\nu_t * f}{t^{1+\alpha}} dt,\ \omega\Big) = (g_\varepsilon * \mathcal{D}_+^\alpha f,\ \omega). \tag{2.31'}$$

By Lemma 0.5 this yields a pointwise equality

$$\int_\varepsilon^\infty \frac{\nu_t * f}{t^{1+\alpha}} dt \overset{a.e.}{=} (k_\varepsilon^+ + h_\varepsilon - \lambda\pi i q_\varepsilon) * \mathcal{D}_+^\alpha f \tag{2.32}$$

which implies the first equality in (2.29) owing to the standard machinery of approximation to the identity (here Lemmas 2.3, 2.4 are important) and the limit properties of the conjugate Poisson integrals (see Neri 1971). Λ

2.3. The case of purely imaginary order.

We need some preliminaries which represent an extension of Lemmas 2.3, 2.4, 2.5 to the case $Re\,\alpha = 0$.

Lemma 2.9. *Let $\nu \in \mathcal{M}$, $Re\,\alpha = 0$. If*

$$\nu(\mathrm{R}) = 0, \quad \int_{|x|>1} |x|^\beta d|\nu|(x) < \infty, \quad \int_{|x|<1} |x|^{-\delta} d|\nu|(x) < \infty \tag{2.33}$$

for some $\beta > 0$ and $\delta \in (0,1)$, then for sufficiently small $\varepsilon_1, \varepsilon_2 > 0$,

$$|\frac{1}{x}(I_+^{1+\alpha}\nu)(x)| \ .\ |x|^{-\varepsilon_1 - 1} \quad \text{for } |x| > 1, \quad \text{and} \quad |k^\pm(x)| \ .\ |x|^{\varepsilon_2 - 1} \quad \text{for } |x| \le 1. \tag{2.34}$$

In particular, $k^+(x) \in L^1$ and $h(x) \in L^1$.

The proof of this lemma is technical and given in Appendix 2.

Lemma 2.10. *Let ν satisfy (2.33) with some $\delta \in (0,1)$, $\beta > 0$ and let $\omega \in \Phi$. Then*

(i) *The relation*

$$\int_{\varepsilon}^{\rho} \frac{(\nu_t, \omega)}{t^{1+\alpha}} dt = (g_\varepsilon, \psi) - (g_\rho, \psi), \quad g = k^+ + h - \lambda \pi i q = k^+ + k^-, \quad \psi = \overline{\mathcal{D}_-^\alpha \bar{\omega}}, \tag{2.35}$$

previously proved for $Re\, \alpha > 0$ (see Lemma 2.5), holds for $-\delta < Re\, \alpha \leq 0$ with the same δ as in (2.33).

(ii) *The relation (2.17) is true for $Re\, \alpha = 0$ with coefficients*

$$\gamma_+ = \begin{cases} \Gamma(-\alpha)[\int_0^\infty x^\alpha d\nu(x) + \cos(\alpha\pi) \int_{-\infty}^0 |x|^\alpha d\nu(x), \ \alpha \neq 0, \\ \int_{-\infty}^{\infty} \log \frac{1}{|x|} d\nu(x), \ \alpha = 0; \end{cases} \tag{2.36}$$

$$\gamma_- = -\frac{\pi i}{\Gamma(1+\alpha)} \int_{-\infty}^0 |x|^\alpha d\nu(x). \tag{2.36$'$}$$

P r o o f. We recall the expressions of k^+, k^-:

$$k^+(x) = (2x)^{-1}[(I_+^{1+\alpha}\nu)(x) - (I_+^{1+\alpha}\nu)(-x)], \tag{2.37}$$

$$k^-(x) = (2x)^{-1}[(I_+^{1+\alpha}\nu)(x) + (I_+^{1+\alpha}\nu)(-x) - 2\lambda], \ \ \lambda = \frac{1}{\Gamma(1+\alpha)} \int_{-\infty}^{\infty} |x|^\alpha d\nu(x). \tag{2.38}$$

In order to prove (2.35), we extend (2.14) to $-\delta < Re\, \alpha \leq 0$ by using analytic continuation. By Lemma 0.6 the left-hand side of (2.35) can be regarded as entire function of α. Let us show that the right-hand side of (2.35) is an analytic function in the strip $\{\alpha \in \mathrm{C} : -\delta < Re\, \alpha < \beta_0\}$ for some $\beta_0 > 0$. For arbitrary fixed $N > 0$ put

$$M_{\delta,\beta,N} = \{\alpha \in \mathrm{C} : -\delta < Re\, \alpha < \beta, \ |Im\, \alpha| < N\},$$

and let $c = c(\delta, \beta, N)$ be a constant which can be different in each occasion. According to (2.35) we have to examine the expressions $(k^+, \psi), (k^-, \psi)$ from the point of view of their analyticity in the α-variable. Consider the first one. Since $k^+(x)$ is even, then by (2.37),

$$(k^+, \psi) = \int_0^\infty \frac{(I_+^{1+\alpha}\nu)(x) - (I_+^{1+\alpha}\nu)(-x)}{2x}[(\mathcal{D}_-^\alpha \bar{\omega})(x) + (\mathcal{D}_-^\alpha \bar{\omega})(-x)]dx =$$

$$= \int_0^\infty R_\alpha(x)\Big[\int_{-\infty}^{-x}[(x-y)^\alpha - (-x-y)^\alpha]d\nu(y) + \int_{-x}^{x}(x-y)^\alpha d\nu(y)\Big]\frac{dx}{x} = I_1 + I_2,$$

$$R_\alpha(x) = \frac{1}{2\Gamma(\alpha+1)}[(\mathcal{D}_-^\alpha \bar{\omega})(x) + (\mathcal{D}_-^\alpha \bar{\omega})(-x)].$$

For I_1 we have

$$I_1 = \int_{-\infty}^{0} |y|^\alpha d\nu(y) \int_0^1 R_\alpha(|y|\xi)[(\xi+1)^\alpha - (1-\xi)^\alpha]\frac{d\xi}{\xi}.$$

Note that

$$(2.39) \qquad (1+x^2)|R_\alpha(x)| \le c \quad \text{for } \alpha \in M_{\delta,\beta,N}.$$

Indeed, since $\omega \in \Phi$, then clearly,

$$|(1+x^2)R_\alpha(x)| \simeq |\int_{-\infty}^{\infty} e^{-i\xi x}(I-\Delta)\hat{R}_\alpha(\xi)d\xi| \lesssim \int_{-\infty}^{\infty} |\xi|^\alpha \; |\psi_1(\xi)|d\xi \le c = c(\delta, \beta, N)$$

(here ψ_1 is a certain continuous function rapidly decreasing at the origin and at infinity). By (2.39), $|R_\alpha(|y|\xi)| \lesssim c$ for $\alpha \in M_{\delta,\beta,N}$. Furthermore,

$$\Big|\frac{(\xi+1)^\alpha - (1-\xi)^\alpha}{\xi}\Big| = \frac{|\alpha|}{\xi}\Big|\int_0^\xi \Big[(1+t)^{\alpha'-1} + (1-t)^{\alpha'-1}\Big]dt\Big| \le$$

$$\le \frac{|\alpha|}{\xi}\int_0^\xi (1 + (1-t)^{-\delta-1})dt \le c\Big(1 + \frac{\xi^{\delta-1}}{(1-\xi)^\delta}\Big).$$

By Lemma 0.6 it follows that the inner integral in I_1 represents an analytic function of $\alpha \in M_{\delta,\beta,N}$ and

$$|y|^\alpha \Big|\int_0^1 R_\alpha(|y|\xi)[(\xi+1)^\alpha - (1-\xi)^\alpha]\frac{d\xi}{\xi}\Big| \le c|y|^\alpha \le c\begin{cases} |y|^{-\delta}, & |y| < 1, \\ |y|^\beta, & |y| > 1. \end{cases}$$

This implies the analyticity of $I_1 = I_1(\alpha)$ for $\alpha \in M_{\delta,\beta,N}$.

In order to prove the analyticity of $I_2 = I_2(\alpha)$ one needs more refined estimates. We have

$$I_2 = \int_0^\infty y^\alpha d\nu(y) \int_1^\infty \frac{R_\alpha(y\xi)(\xi-1)^\alpha}{\xi} d\xi + \int_{-\infty}^0 |y|^\alpha d\nu(y) \int_1^\infty \frac{R_\alpha(|y|\xi)(\xi+1)^\alpha}{\xi} d\xi = A_1 + A_2.$$

By (2.39),

$$|x|^\theta |R_\alpha(x)| \le c = c(\delta,\beta,N,\theta), \quad 0 < \theta \le 2. \tag{2.40}$$

Consider A_1. Take an arbitrary $\delta_0 \in (0,\delta)$ and fix $\beta_0 \in (0, \min\{\delta-\delta_0;\ \beta\})$. Then for $\theta \in ((\beta_0+\delta-\delta_0)/2,\ \delta-\delta_0)$ and $\alpha' = Re\,\alpha \in (-\delta_0, \beta_0)$ by (2.40) we have

$$\Big|R_\alpha(y\xi)\frac{(\xi-1)^\alpha}{\xi}\Big| \le c\frac{(\xi-1)^{\alpha'}}{y^\theta \xi^{1+\theta}} \le \frac{c}{y^\theta}\begin{cases} (\xi-1)^{-\delta_0}, & 1<\xi<2, \\ (\xi-1)^{\beta_0}\xi^{-1-(\beta_0+\delta-\delta_0)/2} \,.\, \xi^{-1-(\delta-\delta_0-\beta_0)/2}, & \xi \ge 2. \end{cases}$$

Owing to Lemma 0.6 this implies the analyticity of the inner integral $\int_1^\infty$ for $\alpha \in M_{\delta_0,\beta_0,N}$. In order to apply Lemma 0.6 again, we note that for α and θ satisfying the above restrictions,

$$\Big|y^\alpha \int_1^\infty \frac{R_\alpha(y\xi)(\xi-1)^\alpha}{\xi} d\xi\Big| \,.\, Cy^{\alpha'-\theta} < C\begin{cases} y^{-\delta}, & y<1, \\ y^\beta, & y \ge 1. \end{cases}$$

By (2.33) it follows that A_1 is an analytic function of α in the domain $-\delta_0 < Re\,\alpha < \beta_0$ which can be arbitrarily close to the line $Re\,\alpha = -\delta$ and contains the imaginary axis. The same holds for A_2. Thus, one can resume that $\alpha \to (k^+, \psi)$ is an analytic function in the strip $-\delta_0 < Re\,\alpha < \beta_0$, that can be chosen arbitrarily close to the line $Re\,\alpha = -\delta$ and contains the axis $Re\,\alpha = 0$. By the same reasoning this is also true for (k^-, ψ) which has the form

$$(k^-, \psi) = \int_0^\infty \frac{(I_+^{1+\alpha}\nu)(x) + (I_+^{1+\alpha}\nu)(-x) - 2\lambda}{2x} [(\mathcal{D}_-^\alpha \bar\omega(x) - (\mathcal{D}_-^\alpha \bar\omega)(-x)]dx =$$

$$= \int_0^\infty N_\alpha(x)\Big[\int_{-x}^x (x-y)^\alpha d\nu(y) - 2\int_{-x}^0 |y|^\alpha d\nu(y)\Big]\frac{dx}{x} + \int_0^\infty N_\alpha(x)\frac{dx}{x}\int_{-\infty}^{-x} [(x+|y|)^\alpha - |y|^\alpha] d\nu(y) +$$

$$+ \int_0^\infty N_\alpha(x)\frac{dx}{x}\int_{-\infty}^{-x} [(|y|-x)^\alpha - |y|^\alpha] d\nu(y), \quad N_\alpha(x) = \frac{1}{2\Gamma(\alpha+1)}[\mathcal{D}_-^\alpha \bar\omega)(x) - (\mathcal{D}_-^\alpha \bar\omega)(-x)].$$

Now we can conclude, that once (2.35) holds in the right neighborhood of the imaginary axis (see Lemma 2.5), then, by analyticity, (2.35) also holds for $-\delta < Re\,\alpha \le 0$.

The proof of (2.15) in the case $Re\,\alpha = 0$ is similar to that in Lemma 2.5 and based on Lemma 2.9 (the reader can easily check by using the Parseval inequality that $(k_\rho^+, \psi) \to 0$, $(h_\rho, \psi) \to 0$, $(q_\rho, \psi) \to 0$ as $\rho \to \infty$). The relations (2.36), (2.36′) are established in Appendix 3. Λ

Now we are ready to prove an analogue of Theorem 2.8 for the case $Re\,\alpha = 0$. First we have to define the expressions $\mathcal{D}_\pm^\alpha f, \mathcal{D}^\alpha f, \mathcal{D}_s^\alpha f$ for $Re\,\alpha = 0$, $f \in L^p$. Operators $\mathcal{D}_\pm^\alpha, \mathcal{D}^\alpha, \mathcal{D}_s^\alpha$ are well determined by (0.9), (0.12) on functions belonging to Φ. On L^p-functions, $1 < p < \infty$, they are understood as the linear bounded (from L^p to L^p) multiplier operators extended from the dense subset Φ.

Theorem 2.11. *Let $Re\,\alpha = 0, \nu \in \mathcal{M}$. Assume that*

$$\nu(\mathrm{R}) = 0, \int\limits_{|x|>1} |x|^\beta d|\nu|(x) < \infty, \int\limits_{|x|<1} |x|^{-\delta} d|\nu|(x) < \infty \tag{2.41}$$

for some $\beta > 0$, $\delta \in (0,1)$, and let $f \in L^p$, $1 < p < \infty$.

Then the limit

$$A^\alpha f = \lim_{\substack{\varepsilon \to 0 \\ \rho \to \infty}} \int\limits_\varepsilon^\rho \frac{\nu_t * f}{t^{1+\alpha}} dt \tag{2.42}$$

exists in the L^p-norm and in the a.e. sense, and the following relations hold:

$$A^\alpha f = \gamma_+ \mathcal{D}_+^\alpha f + \gamma_- H \mathcal{D}_+^\alpha f = c_+ \mathcal{D}_+^\alpha f + c_- \mathcal{D}_-^\alpha f = a_+ \mathcal{D}^\alpha f + a_- \mathcal{D}_s^\alpha f. \tag{2.43}$$

(in the second equality it is assumed $\alpha \neq 0$). Here $\gamma_\pm$ are defined by (2.36), (2.36′), $c_\pm$ have the form (2.20) and $a_\pm$ may be evaluated by the following formulae:

$$a_+ = \begin{cases} \Gamma(-\alpha) \cos \frac{\alpha\pi}{2} \int\limits_{-\infty}^{\infty} |x|^\alpha d\nu(x), & \alpha \neq 0 \\ \int\limits_{-\infty}^{\infty} \log \frac{1}{|x|} d\nu(x), & \alpha = 0. \end{cases} \tag{2.44}$$

$$(2.44') \qquad a_- = \begin{cases} -i\Gamma(-\alpha)\sin\frac{\alpha\pi}{2}\int\limits_{-\infty}^{\infty}|x|^\alpha sgn\, x\; d\nu(x), & \alpha\neq 0\\ \dfrac{\pi i}{2}\int\limits_{-\infty}^{\infty} sgn\, x\; d\nu(x), & \alpha = 0.\end{cases}$$

P r o o f. As in the proof of Theorem 2.8, from (2.35) we get

$$(2.45) \qquad \int\limits_{\varepsilon}^{\rho}\frac{\nu_t * f}{t^{1+\alpha}}dt = g_\varepsilon * \mathcal{D}_+^\alpha f - g_\rho * \mathcal{D}_+^\alpha f, \quad g = k^+ + h - \lambda\pi i q.$$

We prove the first relation in (2.43). Using Lemmas 2.9, 2.10, one can apply well-known results on approximation to the identity (Stein 1970, Neri 1971) and related properties of the conjugate Poisson kernel (Neri 1971). This gives

$$(2.46) \qquad k_\varepsilon^+ * \mathcal{D}_+^\alpha f \to \gamma_+\mathcal{D}_+^\alpha f, \quad h_\varepsilon * \mathcal{D}_+^\alpha f \to 0, \quad q_\varepsilon * \mathcal{D}_+^\alpha f \to H\mathcal{D}_+^\alpha f$$

as $\varepsilon \to 0$ in the L^p-norm and in the a.e. sense. Furthermore, since $k^+,\ h \in L^1$, then $k_\rho^+ * \mathcal{D}_+^\alpha f \overset{L^p}{\to} 0$, $h_\rho * \mathcal{D}_+^\alpha f \overset{L^p}{\to} 0$ as $\rho \to \infty$ (see, e.g., Samko 1984, p. 22). In order to prove that $\lim\limits_{\rho\to\infty}\|q_\rho * \mathcal{D}_+^\alpha f\|_p = 0$ we take the sequence of compactly supported smooth functions ω_n such that $\lim\limits_{m\to\infty}\|\mathcal{D}_+^\alpha f - \omega_m\|_p = 0$, and make use of the uniform estimate $\|\mathcal{D}_+^\alpha f * q_\rho\|_p \le C_p\|\mathcal{D}_+^\alpha f\|_p$ (see Neri 1971). Then

$$\|\mathcal{D}_+^\alpha f * q_\rho\|_p \le \|(\mathcal{D}_+^\alpha f - \omega_m) * q_\rho\|_p + \|\omega_m * q_\rho\|_p \le C_p\|\mathcal{D}_+^\alpha f - \omega_m\|_p + \rho^{\frac{1}{p'}}\|\omega_m\|_1\|q\|_p$$

and the desired L^p-convergence follows.

Let us prove that $g_\rho * \mathcal{D}_+^\alpha f \to 0$ almost everywhere as $\rho \to \infty$. Put $\varphi = \mathcal{D}_+^\alpha f$ and write $g(t)$ in the form:

$$g(t) = \frac{1}{t}(I_+^{1+\alpha}\nu)(t) - \frac{\lambda}{t} = \frac{1}{\Gamma(1+\alpha)t}\Big(\int\limits_0^t (t-y)^\alpha d\nu(y) + \int\limits_{-\infty}^0 [(t+|y|)^\alpha - |y|^\alpha]d\nu(y)\Big) =$$
$$= U^1(t) + U^2(t).$$

We have to show that $\lim\limits_{\rho\to\infty}(U_\rho^i * \varphi)(x) \overset{a.e.}{=} 0$ where $U_\rho^i(t) = \rho^{-1}U^i(t/\rho);\ i = 1,2.$ For any $\theta \in (0,1)$ we have

$$|(U_\rho^i * \varphi)(x)| \le \Big(\int\limits_0^\theta + \int\limits_{-\theta}^0 + \int\limits_\theta^\infty + \int\limits_{-\infty}^{-\theta}\Big)\ |\varphi(x-t)||\ U_\rho^i(t)|dt =$$
$$= N_{\theta,i}^+(x,\rho) + N_{\theta,i}^-(x,\rho) + M_{\theta,i}^+(x,\rho) + M_{\theta,i}^-(x,\rho),\ i = 1,2.$$

If $\rho \geq 1$, then $N^{+}_{\theta,1}(x,\rho)\ .\int\limits_0^\theta |\varphi(x-t)|w(t)dt, \quad w(t)=\dfrac{|\nu|((0,t))}{t}.$

By (2.41), $w \in L^1(0,1)$ and

$$\|\int\limits_0^1 |\varphi(x-t)|w(t)dt\|_p \leq \|\varphi\|_p \int\limits_0^1 w(t)dt < \infty.$$

It means that the function $t \to |\varphi(x-t)|w(t)$ belongs to $L^1(0,1)$ for almost all x and $N^{+}_{\theta,1}(x,\rho)$ becomes arbitrarily small for all sufficiently small $\theta \leq \theta_1(x)$ uniformly in $\rho \geq 1$. The same is true for $N^{-}_{\theta,1}(x,\rho)$. Fix $\theta = \theta_1(x)$ and consider $M^{+}_{\theta_1,1}(x,\rho)$. By Hölder's inequality,

$$\int\limits_{\theta_1}^{\infty} \frac{|\varphi(x-t)|}{t}dt \leq \|\varphi\|_p \Big(\int\limits_{\theta_1}^{\infty} t^{-p'}dt\Big)^{1/p'} < \infty,$$

i.e. the function $t \to |\varphi(x-t)|/t$ belongs to $L^1(\theta_1,\infty)$ for all x. Since for $t > \theta_1$, $|U^1_\rho(t)| \simeq t^{-1}|\nu|((0,t/\rho)) \leq \theta_1^{-1}|\nu|(\mathrm{R})$, then by the Lebesgue dominated convergence theorem

$$\lim_{p\to\infty} M^{+}_{\theta_1,1}(x,\rho)\ .\int\limits_{\theta_1}^{\infty} \frac{|\varphi(x-t)|}{t} \lim_{\rho\to\infty} |\nu|((0,\frac{t}{\rho}))dt = 0.$$

The same is true for $M^{-}_{\theta_1,1}(x,\rho)$. Consider $N^{+}_{\theta,2}(x,\rho)$. Let $0 < \varepsilon < \delta$ with the same δ as in (2.41). Then for $\rho \geq 1$,

$$\left|U^2_\rho(t)\right| \simeq \left|\frac{1}{t}\int\limits_{-\infty}^{0} d\nu(y)\int\limits_0^{t/\rho}(|y|+\xi)^{\alpha-1}d\xi\right|\ .\ \frac{1}{t}\int\limits_{-\infty}^{0}|y|^{-\varepsilon}d|\nu|(y)\int\limits_0^t \xi^{\varepsilon-1}d\xi \simeq t^{\varepsilon-1}.$$

Hence as above, $N^{+}_{\theta,2}$ can be made arbitrarily small for sufficiently small $\theta \leq \theta_2(x)$ uniformly with respect to $\rho \geq 1$. The same is true for $N^{-}_{\theta,2}$. Now fix θ_2 and consider $M^{+}_{\theta_2,2}(x,\rho)$. Since $t \to \dfrac{|\varphi(x-t)|}{t}$ is a summable function on $[\theta_2,\infty)$ for all x, then

$$\lim_{\rho\to\infty} M^{+}_{\theta,2}(x,\rho)\ .\int\limits_{\theta_2}^{\infty}\frac{|\varphi(x-t)|}{t}dt \lim_{\rho\to\infty}\int\limits_{-\infty}^{0} |(\frac{t}{\rho}+|y|)^\alpha - |y|^\alpha|\ d|\nu|(y) = 0.$$

The same is true for $M^{-}_{\theta_2,2}$. This completes the proof of (2.42) and the first equality in (2.43) in the a.e. sense. By (1.6)-(1.7) the expressions in (2.43) coincide in Φ'-sense. By Lemmas 2.6, 2.7, 0.5 these three expressions also coincide almost everywhere. Λ

2.4. Some generalizations.

Let us obtain some analogues of Theorems 2.8, 2.11 in the case when $Re\ \alpha \geq 1$ and ν belongs to a certain class of Φ'-distributions which satisfies the following conditions.

Condition 2.12. *If* $\alpha = \ell + \alpha_0,\ \ell \in \mathrm{N},\ 0 \leq Re\ \alpha_0 < 1,$ *then* $\nu_0 \overset{(\Phi')}{\equiv} I_+^\ell \nu$ *is a finite Borel measure (i.e.* $\nu_0 \in \mathcal{M}$*) such that*

$$\nu_0(\mathrm{R}) = 0, \qquad \int_{|x|>0} |x|^\beta d|\nu_0|(x) < \infty \quad \textit{for some } \beta > Re\ \alpha_0.$$

If $Re\ \alpha_0 = 0$*, we assume additionally, that* $\int_{|x|<1} |x|^{-\delta} d|\nu_0|(x) < \infty$ *for some* $\delta \in (0,1)$*.*

An example of the "Gelfand-Shilov distribution". Let $Re\ \alpha_0 \neq 0$,

$$\nu(x) = \delta(x-1) - \sum_{j=0}^{\ell} \frac{(-1)^j}{j!} \delta^{(j)}(x), \tag{2.47}$$

$\delta(x)$ being the Dirac δ-function. For $\omega \in \Phi$ we have

$$(I_+^\ell[\delta(\cdot - 1) - \sum_{j=0}^{\ell} \frac{(-1)^j}{j!} \delta^{(j)}(\cdot)], \omega) = (I_-^\ell \omega)(1) - \sum_{j=0}^{\ell} \frac{(-1)^j}{j!} (I_-^{\ell-j}\omega)(0) = (\nu_0, \omega)$$

where

$$\nu_0 = \frac{(x-1)_+^{\ell-1}}{\Gamma(\ell)} - \sum_{j=0}^{\ell-1} \frac{(-1)^j x_+^{\ell-j-1}}{j!\Gamma(\ell-j)} + \frac{(-1)^\ell}{\ell!}\delta(x).$$

Clearly, $\mathrm{supp}\, \nu_0 = [0,1]$ and $\nu_0 \in \mathcal{M}$. Moreover, $\nu_0(\mathrm{R}) = \hat{\nu}_0(0) = 0$ because

$$\hat{\nu}_0(\xi) = (-i\xi)^{-\ell}\Big[e^{i\xi} - \sum_{j=0}^{\ell} \frac{(i\xi)^j}{j!}\Big] = (-1)^\ell \sum_{j=\ell+1}^{\infty} \frac{(i\xi)^{j-\ell}}{j!}.$$

Thus, ν satisfies Condition 2.12. Note that for $f \in \Phi$,

$$\int_0^\infty \frac{\nu_t * f}{t^{1+\alpha}} dt = \int_0^\infty \Big[f(x-t) - \sum_{j=0}^{\ell} \frac{(-t)^j}{j!} f^{(j)}(x)\Big] \frac{dt}{t^{1+\alpha}}.$$

This coincides with the Gelfand-Shilov regularization of the divergent integral $\int_0^\infty t^{-\alpha-1} f(x-t)dt$ (cf. Gelfand and Shilov 1964, p. 48).

Theorem 2.13. *Let* $\alpha = \ell + \alpha_0,\ \ \ell \in \mathrm{N},\ \ Re\ \ \alpha_0 \in [0,1)$*. Assume that* ν *satisfies*

Condition 2.12. For $f \in \Phi'$ let one of the derivatives $\mathcal{D}^{\alpha}_{\pm} f$, $\mathcal{D}^{\alpha} f$, $\mathcal{D}^{\alpha}_{s} f$ (in the Φ'-sense) belong to L^p, $1 < p < \infty$. If Re $\alpha \notin$ N, then the integral $\int_{\varepsilon}^{\infty} (\nu_t * f)(x) dt/t^{1+\alpha}$ coincides (in the Φ'-sense) with the L^p-function that tends in the L^p-norm and in the a.e. sense to

$$\gamma_+ \mathcal{D}^{\alpha}_{+} f + \gamma_- H D^{\alpha}_{+} f = c_+ \mathcal{D}^{\alpha}_{+} f + c_- \mathcal{D}^{\alpha}_{-} f = a_+ \mathcal{D}^{\alpha} f + a_- \mathcal{D}^{\alpha}_{s} f.$$

Here the coefficients $\gamma_{\pm}$, $c_{\pm}$, $a_{\pm}$ are defined by equalities (2.36),(2.36′),(2.20) and (2.44),(2.44′) in which α and ν must be replaced by α_0 and $\nu_0 = I^{\ell}_{+} \nu$ respectively. If Re $\alpha \in$ N, then the above statement is valid for the integral $\int_{\varepsilon}^{\rho} (\nu_t * f)(x) dt/t^{1+\alpha}$ with $\varepsilon \to 0$, $\rho \to \infty$.

P r o o f. For $\omega \in \Phi$ denote $\omega_0(x) = (-d/dx)^{\ell} \omega(x)$, $\omega_x(y) = \omega(x+y)$, $\psi_x(y) = \overline{(\mathcal{D}^{\alpha_0}_{-}(\bar{\omega}_0)_x)(y)}$. Then

$$\left(\int_{\varepsilon}^{\infty} \frac{\nu_t * f}{t^{1+\alpha}} dt, \omega \right) = \left(f(x), \overline{\int_{\varepsilon}^{\infty} \frac{(\nu_t, (I^{\ell}_{-} \omega_0)_x)}{t^{1+\alpha}} dt} \right) = \left(f(x), \overline{\int_{\varepsilon}^{\infty} \frac{((\nu_0)_t, (\omega_0)_x)}{t^{1+\alpha_0}} dt} \right)$$
$$= (f(x), \overline{(g_{\varepsilon}, \psi_x)}) = (g_{\varepsilon} * f, \overline{\mathcal{D}^{\alpha_0}_{-} \bar{\omega}_0}) = (g_{\varepsilon} * f, \overline{\mathcal{D}^{\alpha}_{-} \bar{\omega}}) = (g_{\varepsilon} * \mathcal{D}^{\alpha}_{+}, \omega)$$

(cf. the proof of Theorem 2.8). Here $g = k^+ + h - \lambda \pi i q$ is the same as in (2.29′) but with ν replaced by ν_0 and α replaced by α_0. The rest of the proof is the same as in Theorem 2.8. In the case *Re* $\alpha_0 = 0$ (i.e. *Re* $\alpha \in$ N) the argument is similar: one should replace the upper limit ∞ by ρ and make use of the argument in the proof of Theorem 2.11. Λ

3. Wavelet type representation of fractional integrals.

In this section we are concerned with integrals which have the form $I^{\alpha}(\nu, \varphi) = \int_0^{\infty} (\varphi * \nu_t)(x)\, dt/t^{1-\alpha}$, *Re* $\alpha > 0$. As we have seen in Section 2, such integrals, with α replaced by $-\alpha$, represent various fractional derivatives and can serve as solutions to the corresponding integral equations of the first kind. In accordance with this, the operators $I^{\alpha}(\nu, f)$, *Re* $\alpha > 0$, represent fractional integrals (and their linear combinations) and can be regarded as solutions to the corresponding differential or, more generally, pseudo-differential equations.

Given a Φ'-distribution f that agrees with a certain locally integrable function, we shall denote the latter by $[f]_{\Phi}$. In the following it is convenient to discriminate between the cases *Re* $\alpha \in$ N and *Re* $\alpha \notin$ N. In order to make the basic idea clearer we shall not formulate the results in the most general form and leave possible generalizations to the interested reader.

Theorem 3.1 *Let* $\alpha = \ell + \alpha_0$, $\ell \in \mathrm{Z}_+$, $0 \le Re\ \alpha_0 < 1$, $Re\ \alpha > 0$, *and let* ν *be an integrable function with a generalized derivative (in the* Φ'*-sense)* $\nu_0 = (d/dx)^\ell \nu$ *belonging to* $\mathcal{M}$ *and such that*

$$\nu_0(\mathrm{R}) = 0, \qquad \int_{|x|>1} |x|^\beta d|\nu_0| < \infty, \qquad \int_{|x|<1} |x|^{-\delta} d|\nu_0| < \infty,$$

for some $\beta, \delta > 0$. *Assume that for* $\varphi \in L^p$, $1 \le p < \infty$, *one of the* Φ'*-distributions* $I^\alpha_\pm \varphi$, $I^\alpha \varphi$, $I^\alpha_s \varphi$ *agrees with a certain* L^r*-function,* $1 < r < \infty$ *(then the same is true for all the rest). Then*

$$(3.1)\qquad \lim_{\substack{\varepsilon \to 0 \\ \rho \to \infty}} \int_\varepsilon^\rho \frac{\varphi * \nu_t}{t^{1-\alpha}} dt = \tilde{\gamma}_+ [I^\alpha_+ \varphi]_\Phi + \tilde{\gamma}_- H[I^\alpha_+ \varphi]_\Phi =$$

$$(3.2)\qquad = \tilde{c}_+ [I^\alpha_+ \varphi]_\Phi + \tilde{c}_- [I^\alpha_- \varphi]_\Phi \quad (\alpha \notin \mathrm{N})$$

$$(3.3)\qquad = \tilde{a}_+ [I^\alpha \varphi]_\Phi + \tilde{a}_- [I^\alpha_s \varphi]_\Phi$$

(the limit being understood in the L^r*-norm and in the a.e.-sense) where*

$$(3.4)\qquad \tilde{\gamma}_+ = \begin{cases} \Gamma(\alpha_0) \left[\int_0^\infty x^{-\alpha_0} d\nu_0(x) + \cos(\alpha_0 \pi) \int_{-\infty}^0 x^{-\alpha_0} d\nu_0(x) \right], & \alpha_0 \ne 0, \\ \int_{-\infty}^\infty \log \frac{1}{|x|} d\nu_0(x), & \alpha_0 = 0, \end{cases}$$

$$(3.5)\qquad \tilde{\gamma}_- = -\frac{\pi i}{\Gamma(1-\alpha_0)} \int_{-\infty}^0 |x|^{-\alpha} d\nu_0(x);$$

$$(3.6)\qquad \tilde{c}_+ = \Gamma(\alpha_0) \int_0^\infty x^{-\alpha_0} d\nu_0(x), \qquad \tilde{c}_- = (-1)^\ell \Gamma(\alpha_0) \int_{-\infty}^0 |x|^{-\alpha_0} d\nu_0(x);$$

$$(3.7)\qquad \tilde{a}_+ = (\tilde{c}_+ + \tilde{c}_-) \cos(\alpha\pi/2), \qquad \tilde{a}_- = i(\tilde{c}_+ - \tilde{c}_-) \sin(\alpha\pi/2), \quad \alpha \notin \mathrm{N},$$

$$(3.8)\qquad \tilde{a}_+ = \begin{cases} (-1)^k \int_{-\infty}^\infty \log \frac{1}{|x|} d\nu_0, & \alpha = 2k, \\ (-1)^{k-1} \pi \int_{-\infty}^0 d\nu_0, & \alpha = 2k-1, \quad k \in \mathrm{N}, \end{cases}$$

$$\tilde{a}_- = \begin{cases} (-1)^{k+1}\pi i \int_{-\infty}^{0} d\nu_0, & \alpha = 2k, \\ (-1)^{k-1} i \int_{-\infty}^{\infty} \log \frac{1}{|x|} d\nu_0, & \alpha = 2k-1, \quad k \in \mathrm{N}. \end{cases} \tag{3.9}$$

P r o o f. For $\omega \in \Phi$, denote $\omega_x(y) = \omega(x+y)$, $\omega_0 = I_-^\ell \omega$, $\psi_x(y) = \overline{(I_-^{\alpha_0}(\bar{\omega}_0)_x)}(y) = \overline{(I_-^{\alpha_0}\bar{\omega}_0)}(x+y)$. Then, by making use of (2.35), we have

$$\left(\int_\varepsilon^\rho \frac{\varphi * \nu_t}{t^{1-\alpha}} dt, \omega\right) = \left(\varphi(x), \overline{\int_\varepsilon^\rho \frac{(\nu_t, (-1)^\ell (\omega_0^{(\ell)})_x)}{t^{1-\alpha}} dt}\right)$$
$$= \left(\varphi(x), \overline{\int_\varepsilon^\rho \frac{((\nu_0)_t, (\omega_0)_x)}{t^{1-\alpha_0}} dt}\right) = (\varphi(x), \overline{(g_\varepsilon - g_\rho, \psi_x)})$$
$$= ((g_\varepsilon - g_\rho) * \varphi, \overline{I_-^{\alpha_0}\bar{\omega}_0}) = ((g_\varepsilon - g_\rho) * \varphi, \overline{I_-^{\alpha}\bar{\omega}}) = ((g_\varepsilon - g_\rho) * [I_+^\alpha \varphi]_\Phi, \omega)$$

(cf. the proofs of Theorems 2.8, 2.11). Here $g = k^+ + h - \lambda\pi i q$ is the same function as in (2.35) but with α and ν replaced by $-\alpha_0$ and ν_0 respectively. By Lemma 0.5 this leads to the pointwise equality

$$\int_\varepsilon^\rho \frac{\nu_t * \varphi}{t^{1-\alpha}} dt = (g_\varepsilon - g_\rho) * [I_+^\alpha \varphi]_\Phi. \tag{3.10}$$

As in the proof of Theorem 2.11, the right-hand side of (3.10) tends (in the required sense) to $\tilde{\gamma}_+[I_+^\alpha\varphi]_\Phi + \tilde{\gamma}_- H[I_+^\alpha \varphi]_\Phi$ with $\tilde{\gamma}_\pm$ defined by (3.4), (3.5). The relations (3.2), (3.3) follow from (3.1) by (1.6)-(1.9). Λ

Consider the case $Re\,\alpha \notin \mathrm{N}$.

Lemma 3.2. *Let $\alpha = \ell + \alpha_0$, $\ell \in \mathrm{Z}_+$, $0 < Re\,\alpha_0 < 1$. Assume that ν is a Φ'-distribution such that $\nu_0 = (d/dx)^\ell \nu$ is a measure. Put*

$$k^\pm(x) = (\pm 1)^\ell |x|^{-1} (I_{0\pm}^{1-\alpha_0}\nu_0)(x) \quad \textit{if} \quad \pm x > 0 \quad \textit{and} \quad k^\pm(x) = 0 \quad \textit{if} \quad \pm x < 0.$$

If

$$\int_{-\infty}^{\infty} |x|^{-Re\,\alpha_0} d|\nu_0|(x) < \infty, \tag{3.11}$$

then $k^\pm \in L^1$, and for any $\omega \in \Phi$,

$$\int_\varepsilon^\rho \frac{(\nu_t, \omega)}{t^{1-\alpha}} dt = \left(k_\varepsilon^+ - k_\rho^+, \overline{I_-^\alpha \bar{\omega}}\right) + \left(k_\varepsilon^- - k_\rho^-, \overline{I_+^\alpha \bar{\omega}}\right). \tag{3.12}$$

If $\varepsilon \to 0, \ \rho \to 0$, *then*

$$(3.13) \qquad \int_\varepsilon^\rho \frac{(\nu_t,\omega)}{t^{1-\alpha}} dt \to \tilde{c}_+ h_+^\alpha + \tilde{c}_- h_-^\alpha, \quad \tilde{c}_\pm = (\pm 1)^\ell \Gamma(\alpha_0) \int_{R_\pm} |x|^{-\alpha_0} d\nu_0(x).$$

P r o o f. Let first $0 < Re \ \alpha < 1$, i.e. $\ell = 0, \ \alpha_0 = \alpha, \ \nu_0 = \nu$. The relations $k^\pm \in L^1$ and $\int k^\pm = \tilde{c}_\pm$ follow easily from (3.11) by changing the order of integration. Then (3.13) is implied by (3.12). In order to prove (3.12) we have

$$\int_\varepsilon^\rho \frac{(\nu_t,\omega)}{t^{1-\alpha}} dt = \int_0^\infty x^{-\alpha} d\nu(x) \int_{\varepsilon x}^{\rho x} \frac{\bar{\omega}(y)}{y^{1-\alpha}} dy + \int_{-\infty}^0 |x|^{-\alpha} d\nu(x) \int_{\varepsilon|x|}^{\rho|x|} \frac{\bar{\omega}(-y)}{y^{1-\alpha}} dy$$
$$= A_\varepsilon - A_\rho,$$
$$A_\sigma = \int_0^\infty x^{-\alpha} d\nu(x) \int_{\sigma x}^\infty \frac{\bar{\omega}(y)dy}{y^{1-\alpha}} + \int_{-\infty}^0 |x|^{-\alpha} d\nu(x) \int_{-\infty}^{\sigma x} \frac{\bar{\omega}(y)dy}{y^{1-\alpha}},$$

$\sigma = \varepsilon, \rho$. If we replace $x^{-\alpha} y^{\alpha-1}$ and $|x|^{-\alpha}|y|^{\alpha-1}$ according to the formula

$$a^{-\alpha} b^{\alpha-1} = \frac{\sigma^\alpha}{\Gamma(\alpha)\Gamma(1-\alpha)} \int_{\sigma a}^b (b-t)^{\alpha-1}(t-\sigma a)^{-\alpha} \frac{dt}{t}, \quad a, b > 0.$$

(see Gradshteyn and Ryzhik 1980 , formula 3.228(2)), then simple calculations will lead to (3.12). If $\ell \geq 1$, then

$$\int_\varepsilon^\rho \frac{(\nu_t,\omega)}{t^{1-\alpha}} dt = \int_\varepsilon^\rho \frac{((\nu_t,(-d/dx)^\ell I_-^\ell \omega)}{t^{1-\alpha}} dt = \int_\varepsilon^\rho \frac{((\nu_0)_t, I_-^\ell \omega)}{t^{1-\alpha_0}} dt$$
$$= (\overset{\circ}{k}{}_\varepsilon^+ - \overset{\circ}{k}{}_\rho^+, I_-^{\alpha_0} I_-^\ell \omega) + (\overset{\circ}{k}{}_\varepsilon^- - \overset{\circ}{k}{}_\rho^-, I_+^{\alpha_0} I_-^\ell \omega), \quad \overset{\circ}{k}{}^\pm(x) = (\pm 1)^\ell k^\pm(x).$$

Since $I_-^\ell \omega = (-1)^\ell I_+^\ell \omega$ the result then follows. Λ

Theorem 3.3. *Let* $\alpha = \ell + \alpha_0, \ \ell \in Z_+, \ 0 < Re \ \alpha_0 < 1$. *Assume that* ν *is a finite measure such that* $\nu_0 = (d/dx)^\ell \nu$ *(the derivative is understood in the* Φ'*-sense) is a measure satisfying (3.11). Let* $\varphi \in L^p, \ 1 \leq p < \infty$, *and suppose that one of the* Φ'*-distributions* $I_\pm^\alpha \varphi, \ I^\alpha \varphi, \ I_s^\alpha \varphi$ *agrees with a certain* L^r*-function,* $1 < r < \infty$. *Then*

$$(3.14) \qquad \lim_{\rho \to \infty} \int_0^\rho \frac{\varphi * \nu_t}{t^{1-\alpha}} dt = \tilde{c}_+ [I_+^\alpha \varphi]_\Phi + \tilde{c}_- [I_-^\alpha \varphi]_\Phi = \tilde{a}_+ [I^\alpha \varphi]_\Phi + \tilde{a}_- [I_s^\alpha \varphi]_\Phi,$$

where the limit is interpreted in the L^r-norm and the coefficients $\tilde{c}_\pm$, $\tilde{a}_\pm$ are defined by (3.6), (3.7).

P r o o f. As in Theorem 3.1, it suffices to establish the first relation. Let $\omega \in \Phi$, $\omega_x(y) = \omega(x+y)$. Then

$$\left(\int_0^\rho \frac{\varphi * \nu_t}{t^{1-\alpha}} dt, \omega \right) = \left(\varphi(x), \overline{\int_0^\rho \frac{(\nu_t, \omega)}{t^{1-\alpha}} dt} \right) = (\varphi, \bar{u}_\rho)$$

where, according to (3.12),

$$\begin{aligned} u_\rho(x) &= \lim_{\varepsilon \to 0} \int_\varepsilon^\rho \frac{(\nu_t, \omega_x)}{t^{1-\alpha}} dt \\ &= \lim_{\varepsilon \to 0} \left[(k_\varepsilon^+ - k_\rho^+, \overline{I_-^\alpha \bar{\omega}_x}) + (k_\varepsilon^- - k_\rho^-, \overline{I_+^\alpha \bar{\omega}_x}) \right] \\ &= \tilde{c}_+ (I_-^\alpha \bar{\omega})(x) + \tilde{c}_- (I_+^\alpha \bar{\omega})(x) - (k_\rho^+, \overline{I_-^\alpha \bar{\omega}_x}) - (k_\rho^-, \overline{I_+^\alpha \bar{\omega}_x}). \end{aligned}$$

Thus,

$$\left(\int_0^\rho \frac{\varphi * \nu_t}{t^{1-\alpha}} dt, \omega \right) = (\tilde{c}_+ I_+^\alpha \varphi + \tilde{c}_- I_-^\alpha \varphi - k_\rho^+ * I_+^\alpha \varphi - k_\rho^- * I_+^\alpha \varphi, \omega).$$

By Lemma 0.5,

$$\int_0^\rho \frac{\varphi * \nu_t}{t^{1-\alpha}} dt \overset{a.e.}{=} \tilde{c}_+ [I_+^\alpha \varphi]_\Phi + \tilde{c}_- [I_-^\alpha \varphi]_\Phi - k_\rho^+ * [I_+^\alpha \varphi]_\Phi - k_\rho^- * [I_+^\alpha \varphi]_\Phi$$

where the last two terms tend to 0 in the L^r-norm as $\rho \to \infty$ (use Theorem 2.11). Λ

4. Examples.

In this section we consider wavelet type representations of some classical operators in the fractional calculus.

4.1. Representation of fractional derivatives $\mathcal{D}_+^\alpha f$, $Re\, \alpha > 0$.

Theorem 4.1. *Let ν be a finite Borel measure on R satisfying the following conditions:*

a) $\displaystyle\int_{-\infty}^{\infty} x^j d\nu(x) = 0 \quad \forall\, j = 0, 1, 2, \ldots\, l;$ $\displaystyle\int_{|x|>1} |x|^\beta d|\nu|(x) < \infty$ *for some* $\beta > Re\, \alpha$,

$$b) \int_{-\infty}^{0} |x|^\alpha d\nu(x) = 0, \qquad \gamma_+ = \left\{ \begin{array}{ll} \Gamma(-\alpha) \int_0^\infty x^\alpha d\nu(x), & \alpha \notin \mathrm{N} \\ \frac{(-1)^\ell}{\ell!} \int_{-\infty}^{\infty} x^\ell \log \frac{1}{|x|} d\nu(x), & \alpha = \ell \in \mathrm{N} \end{array} \right\} \neq 0$$

(for instance, one can take ν supported by $[0,\infty]$). If $f \in L^r$, $1 \le r < \infty$, and the derivative $D^\alpha f$ (in the Φ'-sense) belongs to L^p, $1 < p < \infty$, then

$$D^\alpha f = \frac{1}{\gamma_+} \int_0^\infty \frac{\nu_t * f}{t^{1+\alpha}} dt \equiv \lim_{\varepsilon \to 0} \frac{1}{\gamma_+} \int_\varepsilon^\infty \frac{\nu_t * f}{t^{1+\alpha}} dt$$

where the limit exists in the L^p-norm and in the a.e. sense.

This statement follows from Theorem 2.8.

Remark 4.2. In Theorem 4.1 one can take the following finite Borel measure (see Rubin 1994)

$$\nu = \frac{1}{d_k} \sum_{j=0}^{k} c_{j,k} \delta_{\lambda_j}, \qquad k > Re\, \alpha,$$

where δ_{λ_j} is the Dirac measure, concentrated at the point $\lambda_j \in \mathrm{R}$, $0 < \lambda_0 < \ldots < \lambda_k$, $d_k = \prod_{k>i>j\ge 1} (\lambda_i - \lambda_j)$ and $c_{j,k}$ are defined in such a way that

$$\nu_t * f \equiv (\Delta_t^k f)(x) \equiv \frac{1}{d_k} \begin{vmatrix} f(x-\lambda_0 t) & 1 & \lambda_0 & \ldots & \lambda_0^{k-1} \\ f(x-\lambda_1 t) & 1 & \lambda_1 & \ldots & \lambda_1^{k-1} \\ & & \ldots & & \\ f(x-\lambda_k t) & 1 & \lambda_k & \ldots & \lambda_k^{k-1} \end{vmatrix} \equiv \frac{1}{d_k} \sum_{j=0}^{k} c_{j,k} f(x-\lambda_j t).$$

Remark 4.3. Similarly from Theorem 2.8 one can obtain wavelet type representations of the Riesz derivatives $D^\alpha f$, $Re\, \alpha > 0$, and the conjugate Riesz derivatives $D_s^\alpha f$.

4.2. Inversion of Feller potentials.

Consider the integral operator

$$M^\alpha \varphi = \frac{1}{\Gamma(\alpha)} \int_{-\infty}^{\infty} \frac{c_1 + c_2 \,\mathrm{sgn}(x-y)}{|x-y|^{1-\alpha}} \varphi(y) dy, \qquad 0 < Re\, \alpha < 1,$$

which is known as the Feller potential (see Samko, Kilbas and Marichev 1993, p. 214).

The following statement, which follows from Theorem 2.8, enables us to obtain a wavelet type representation for the solution of the integral equation $M^\alpha \varphi = f$ provided that $A = 4(c_1^2 \cos^2 \frac{\alpha\pi}{2} + c_2^2 \sin^2 \frac{\alpha\pi}{2}) \neq 0$.

Theorem 4.4. *Let* $\nu_1, \nu_2 \in \mathcal{M}$ *and let* $c_{\pm}^{(i)}$ *be defined by*

$$c_+^{(i)} = \Gamma(-\alpha) \int\limits_0^{\infty} x^{\alpha} d\nu_i(x), \qquad c_-^{(i)} = \Gamma(-\alpha) \int\limits_{-\infty}^{0} |x|^{\alpha} d\nu_i(x), \qquad i = 1, 2.$$

Assume that ν_i *satisfy the following conditions:*

$$\int\limits_{-\infty}^{\infty} d\nu_i(x) = 0, \qquad \int\limits_{|x|>1} |x|^{\beta} d|\nu_i|(x) < \infty \qquad \text{for some } \beta > Re\,\alpha, \quad i = 1, 2\,;$$

ν_1 *is such that* $c_+^{(1)} = 1$, $c_-^{(1)} = 0$, *and* ν_2 *is such that* $c_+^{(2)} = 0$, $c_-^{(2)} = 1$. *If the equation* $M^{\alpha}\varphi = f$, $f \in L^r$, $1 < r < \infty$, *has a solution* $\varphi \in L^p$, $1 < p < 1/Re\,\alpha$, *then for* $\nu = \dfrac{(c_1 + c_2)\nu_1 + (c_1 - c_2)\nu_2}{A}$ *we have*

$$\varphi = \tfrac{1}{A}((c_1 + c_2)\mathcal{D}_+^{\alpha} f + (c_1 - c_2)\mathcal{D}_-^{\alpha} f) = \lim_{\varepsilon \to 0} \int\limits_{\varepsilon}^{\infty} \frac{\nu_t * f}{t^{1+\alpha}} dt,$$

where the limit is understood in the L^p*-norm and the a.e. sense.*

4.3. Representation of fractional integrals.

Example 4.5. Consider the fractional integral $I_+^{\alpha}\varphi$ for $0 < \alpha < 1$, $1 \le p < \infty$. For $\alpha \ge 1/p$ this integral can be regarded as the Φ'-distribution. Assume additionally, that there is a function $f \in L^r$, $1 < r < \infty$, such that $I_+^{\alpha}\varphi = f$ in the Φ'-sense. Let $\nu = \delta_1$ be the unit Dirac mass at the point $x = 1$. Clearly, ν is a finite measure satisfying (3.11), and $(\varphi * \nu_t)(x) = \varphi(x - t)$. Then, by (3.6), $\tilde{c}_+ = \Gamma(\alpha)$, $\tilde{c}_- = 0$, and we have

$$\text{(4.1)} \qquad \overset{(L^r)}{\lim_{\rho \to \infty}} \frac{1}{\Gamma(\alpha)} \int_0^{\rho} \frac{\varphi(x - t)}{t^{1-\alpha}} dt = f(x).$$

For example, let f belong to the Sobolev space L_{α}^p, $1 < p < \infty$, $0 < \alpha < 1$. Then $f \in L^p$ and there exists $\varphi \in L^p$ such that $f = I_+^{\alpha}\varphi$ in the Φ'-sense. According to Theorem 3.3, f can be represented by (3.14) with $r = p$.

Example 4.6. Let $f \in L^r$, $\varphi \in L^p$, $0 < \alpha < 1$, $1 \le p < \infty$, $1 < r < \infty$. Assume that $f = I^{\alpha}\varphi$ in the Φ'-sense. Take $\nu = \delta_1 + \delta_{-1}$. Then for $\rho \to \infty$,

$$\int_0^{\rho} \frac{\varphi * \nu_t}{t^{1-\alpha}} dt = \int_{-\rho}^{\rho} \frac{\varphi(x - t)}{|t|^{1-\alpha}} dt \xrightarrow{L^r} \tilde{a}_+ f, \quad \tilde{a}_+ = 2\Gamma(\alpha)\cos(\alpha\pi/2) \quad \text{(cf. (3.6), (3.7))}.$$

Example 4.7 (application to simplest differential equations).

Let ν, $\tilde{\gamma}_\pm$ be the same as in Theorem 3.1. Assume that $\varphi \in L^p$, $1 < p < \infty$, and consider the simplest differential equation

$$f' = \varphi \quad \text{(in the } \Phi'\text{-sense).} \tag{4.2}$$

If there exists a solution f of (4.2) belonging to L^r, $1 < r < \infty$, then f may be defined as the L^r-limit:

$$f = \lim_{\substack{\rho\to\infty \\ \varepsilon\to 0}} \frac{1}{\tilde{\gamma}_+} \int_\varepsilon^\rho (\nu_t * \varphi) dt.$$

This assertion follows from Theorem 3.1. The reader can easily generalize it to simplest differential equations $f^{(k)} = \varphi$ of higher orders.

Appendix 1

P r o o f o f L e m m a 2 . 3 . Let us estimate $\frac{1}{x}(I_+^{1+\alpha}\nu)(x)$ for $|x| > 1$. Put $\alpha' = Re\,\alpha$. If $x < -1$, then

$$\left|\frac{1}{x}(I_+^{1+\alpha}\nu)(x)\right| \,.\, \frac{1}{|x|} \int_{-\infty}^{x} (|y| - |x|)^{\alpha'} d|\nu|(y) \leq \frac{1}{|x|} \int_{-\infty}^{x} |y|^{\alpha'} d|\nu|(y) \,.\, |x|^{\alpha'-\beta-1}.$$

Consider the case $x > 1$. By Taylor's formula,

$$\frac{1}{x}(I_+^{1+\alpha}\nu)(x) = \frac{1}{x} \int_{-\infty}^{x} \Big[\sum_{j=0}^{m} \frac{(-y)^j x^{\alpha-j}}{j!\Gamma(1+\alpha-j)} + \frac{(-1)^{m+1}}{m!\Gamma(\alpha-m)} \int_0^y (y-t)^m (x-t)^{\alpha-m-1}\Big] d\nu(y)$$

$$= \sum_{j=0}^{m} A_j + B, \;\; m = [\alpha']$$

(for $\alpha = m$ the term B disappears). The relation (2.9) yields

$$|A_j| \,.\, x^{\alpha'-j-1} \int_x^\infty y^j d|\nu|(y) \leq x^{\alpha'-\beta-1} \int_x^\infty y^\beta d|\nu|(y) \,.\, x^{\alpha'-\beta-1}.$$

For the term B we have

$$|B| \lesssim \Big(\int_{-\infty}^{0} + \int_{0}^{x}\Big)\Big|\int_{0}^{y}(y-t)^m(x-t)^{\alpha-m-1}dt\Big|d|\nu|(y) = B_1 + B_2,$$

where

$$B_1 \le \frac{1}{x}\int_{-\infty}^{0} d|\nu|(y)\int_{0}^{|y|}(|y|-t)^m(x+t)^{\alpha'-m-1}dt = \frac{1}{x}\int_{0}^{\infty}(x+t)^{\alpha'-m-1}dt\int_{-\infty}^{-t}(|y|-t)^m d|\nu|(y)$$

$$\lesssim \frac{1}{x}\sum_{j=0}^{m}\int_{0}^{\infty}(x+t)^{\alpha'-m-1}t^{m-j}dt\int_{-\infty}^{-t}|y|^j d|\nu|(y) \le \frac{1}{x}\int_{0}^{\infty}(x+t)^{\alpha'-m-1}t^{m-\beta}dt\int_{-\infty}^{-t}|y|^\beta d|\nu|(y)$$

$$\le \frac{1}{x}\int_{0}^{\infty}(x+t)^{\alpha'-m-1}t^{m-\beta}dt\Big(\int_{-\infty}^{-1}|y|^\beta d|\nu|(y) + \int_{-1}^{0}d|\nu|(y)\Big) \simeq x^{\alpha'-\beta-1}.$$

Let us estimate B_2:

$$B_2 = \frac{1}{x}\Big(\int_{0}^{x/2} + \int_{x/2}^{x}\Big)d|\nu|(y)\int_{0}^{y}(y-t)^m(x-t)^{\alpha'-m-1}dt = B_{21} + B_{22},$$

where

$$B_{21} \le x^{\alpha'-m-2}\int_{0}^{x/2}y^{m+1}d|\nu|(y) \lesssim x^{\alpha'-m-2}\Big(\int_{0}^{1/2}d|\nu|(y) + \int_{1/2}^{x/2}y^{m+1}d|\nu|(y)\Big) \lesssim x^{\alpha'-1-\min(\beta,m+1)}.$$

$$B_{22} \le \frac{1}{x}\int_{x/2}^{x}d|\nu|(y)\int_{0}^{y}(y-t)^{\alpha'-1}dt \simeq \frac{1}{x}\int_{x/2}^{x}y^{\alpha'}d|\nu|(y) \simeq x^{\alpha'-\beta-1}.$$

Thus, the first relation in (2.10) is proved. Let us estimate $k^+(x)$ for $|x|<1$. Since k^+ is even, it suffices to consider the case $0<x<1$. We have (see Lemma 2.2)

$$2xk^+(x) = \frac{1}{\Gamma(\alpha+1)}\Big(\int_{-x}^{x}(x-y)^\alpha d\nu(y) + \int_{-\infty}^{-x}[(x-y)^\alpha - (-x-y)^\alpha]d\nu(y)\Big) = I_1 + I_2,$$

where

$$|I_1| \lesssim x^{\alpha'}\int_{-x}^{x}d|\nu|(y) \lesssim x^{\alpha'},$$

and for sufficiently small $\varepsilon > 0$,

$$|I_2| \;.\; \int_{-\infty}^{-x} d|\nu|(y) \int_{-x}^{x} (t+|y|)^{\alpha'-1} dt = \int_{-\infty}^{-x} d|\nu|(y) \int_{-x}^{x} (t+|y|)^{\varepsilon-1}(t+|y|)^{\alpha'-\varepsilon} dt \leq$$

$$\leq \int_{-\infty}^{-x} (1+|y|)^{\alpha'-\varepsilon} d|\nu|(y) \left(x^{\varepsilon} + \int_{0}^{x} \frac{d\tau}{(|y|-\tau)^{1-\varepsilon}} \right) \leq$$

$$\leq \int_{-\infty}^{-x} (1+|y|)^{\alpha'-\varepsilon} d|\nu|(y) \left(x^{\varepsilon} + \int_{0}^{x} \frac{d\tau}{(x-\tau)^{1-\varepsilon}} \right) \;.\; x^{\varepsilon}.$$

This gives the required estimate.

Let us estimate $k^{-}(x)$ for $|x| < 1$. By taking into account that k^{-} is odd we consider only the case $0 < x < 1$. We have

$$2xk^{-}(x) = J_1 + J_2$$

where $$|J_1| = \left| \frac{1}{\Gamma(\alpha+1)} \int_{-x}^{x} (x - y)^{\alpha} d\nu(y) - \frac{2}{\Gamma(\alpha+1)} \int_{-x}^{0} |y|^{\alpha} d\nu(y) \right| \;.\; x^{\alpha'}$$

and $$|J_2| = \left| \int_{-\infty}^{-x} [(x-y)^{\alpha} + (-x-y)^{\alpha} - 2|y|^{\alpha}] d\nu(y) \right| .$$

$$.\; \int_{-\infty}^{-x} d|\nu|(y) \int_{0}^{x} (t+|y|)^{\alpha'-1} dt + \int_{-\infty}^{-x} d|\nu|(y) \int_{-x}^{0} (t+|y|)^{\alpha'-1} dt.$$

If $\alpha' < 1$, then $$|J_2| \;.\; \int_{-\infty}^{-x} d|\nu|(y) \int_{0}^{x} t^{\alpha'-1} dt + \int_{-\infty}^{-x} d|\nu|(y) \int_{-x}^{0} (t+x)^{\alpha'-1} dt \;.\; x^{\alpha'}.$$

If $\alpha' \geq 1$, then $$|J_2| \leq x\Big(\int_{-\infty}^{-x} (2|y|)^{\alpha'-1} d|\nu|(y) + \int_{-\infty}^{-x} |y|^{\alpha'-1} d|\nu|(y) \Big) \;.\; x.$$

The proof is complete. Λ

Appendix 2

P r o o f o f L e m m a 2 . 9. Let us estimate $\frac{1}{x}(I_{+}^{1+\alpha}\nu)(x)$. Consider the case

$x > 1$. For $\alpha \neq 0$ we have

$$\left|(I_+^{1+\alpha}\nu)(x)\right| \,.\, \left|\int_{-\infty}^{-x} (x-y)^\alpha d\nu(y)\right| + \left|\int_{-x}^{x} (x-y)^\alpha d\nu(y)\right| = I_1 + I_2,$$

where

$$I_1 \leq \int_{-\infty}^{-x} |y|^\beta |y|^{-\beta} d|\nu|(y) \leq x^{-\beta} \int_{-\infty}^{-x} |y|^\beta d|\nu|(y) \,.\, x^{-\beta}.$$

The estimation of I_2 is more complicated. By (2.33) we obtain:

$$I_2 \leq \left|\int_{-x}^{x} [(x-y)^\alpha - x^\alpha] d\nu(y)\right| + \left|x^\alpha \int_{-\infty}^{-x} d\nu(y) + x^\alpha \int_{x}^{\infty} d\nu(y)\right| \,.$$

$$.\, \left|\int_{-x}^{x} [(x-y)^\alpha - x^\alpha] d\nu(y)\right| + x^{-\beta} \,.\, x^{-\beta} + \left|\int_{-x}^{-x/2} [(x-y)^\alpha - x^\alpha] d\nu(y)\right| +$$

$$+ \left|\int_{-x/2}^{x/2} d\nu(y) \int_{0}^{y} (x-t)^{\alpha-1} dt\right| + \left|\int_{x/2}^{x} [(x-y)^\alpha - x^\alpha] d\nu(y)\right| = x^{-\beta} + A_1 + A_2 + A_3.$$

Consider A_3 (the estimate for A_1 is similar):

$$A_3 \,.\, \int_{x/2}^{x} y^\beta y^{-\beta} d|\nu|(y) \,.\, x^{-\beta}.$$

Let us estimate A_2.

$$A_2 \leq \left|\int_{0}^{x/2} d\nu(y) \int_{0}^{y} (x-t)^{\alpha-1} dt\right| + \left|\int_{-x/2}^{0} d\nu(y) \int_{0}^{y} (x-t)^{\alpha-1} dt\right| = A_2^1 + A_2^2.$$

Consider A_2^1 (the estimate for A_2^2 is similar). For sufficiently small $\varepsilon > 0$ we have

$$A_2^1 = \left|\int_{0}^{x/2} d\nu(y) \int_{0}^{y} t^{1-\varepsilon} t^{\varepsilon-1} (x-t)^{\alpha-1} dt\right| \leq \int_{0}^{x/2} y^{1-\varepsilon} d|\nu|(y) \int_{0}^{y} \frac{t^{\varepsilon-1}}{x-t} dt.$$

Since $x - t \geq x - y \geq \frac{x}{2}$, then

$$A_2^1 \,.\, \frac{1}{x} \int_{0}^{x/2} y^{1-\varepsilon} d|\nu|(y) \int_{0}^{y} t^{\varepsilon-1} dt \,.\, \frac{1}{x^\varepsilon} \int_{0}^{x/2} y^\varepsilon d|\nu|(y) \,.\, \frac{1}{x^\varepsilon}.$$

Thus, for $x > 1$, $\alpha \neq 0$ we get $\dfrac{1}{x}(I_+^{1+\alpha}\nu)(x) = O(x^{-\varepsilon-1})$ for sufficiently small $\varepsilon > 0$. If $x > 1$, $\alpha = 0$, then

$$\left|(I_+^{1+\alpha}\nu)(x)\right| = \Big|\int_{-\infty}^{x} d\nu(y)\Big| = \Big|\int_{x}^{\infty} d\nu(y)\Big| \leq x^{-\beta}.$$

In the case $x < -1$ the estimation of $\dfrac{1}{x}(I_+^{1+\alpha}\nu)(x)$ is trivial by taking into account that

$$\Big(I_+^{1+\alpha}\nu)(x)\Big| \,.\, \Big|\int_{-\infty}^{x} (x-y)^{\alpha} d\nu(y)\Big| \leq |x|^{-\beta} \int_{-\infty}^{x} |y|^{\beta} d|\nu|(y) \,.\, |x|^{-\beta}.$$

Let us estimate $k^+(x)$ for $|x| < 1$. Since k^+ is even, it suffices to consider the case $0 < x < 1$. We have

$$|2xk^+(x)| \,.\, \Big|\int_{-x}^{x} (x-y)^{\alpha} d\nu(y)\Big| + \Big|\int_{-\infty}^{-x} [(x-y)^{\alpha} - (x-y)^{\alpha}] d\nu(y)\Big| = F_1 + F_2.$$

By (2.33),

$$\text{(A2.1)} \qquad |F_1| \leq x^{\delta} \int_{-x}^{x} |y|^{-\delta} d|\nu|(y) \,.\, x^{\delta}.$$

Consider F_2:

$$\text{(A2.2)} \qquad F_2 \,.\, \Big|\int_{-\infty}^{-x} d\nu(y) \int_{0}^{x} (|y|+t)^{\alpha-1} dt\Big| + \Big|\int_{-\infty}^{-x} d\nu(y) \int_{0}^{x} (|y|-t)^{\alpha-1} dt\Big| = B_1 + B_2.$$

For sufficiently small $\varepsilon > 0$ we have

$$B_1 \leq \int_{-\infty}^{-x} |y|^{-\varepsilon} d|\nu|(y) \int_{0}^{x} (|y|+t)^{\varepsilon-1} dt \leq \int_{-\infty}^{-x} |y|^{-\varepsilon} d|\nu|(y) \int_{0}^{x} t^{\varepsilon-1} dt \simeq x^{\varepsilon}.$$

Let us estimate B_2:

$$B_2 \leq \Big|\int_{-x-\sqrt{x}}^{-x} [(|y|-x)^{\alpha} - |y|^{\alpha}] d\nu(y)\Big| + \Big|\int_{-\infty}^{-x-\sqrt{x}} d\nu(y) \int_{0}^{x} (|y|-t)^{\alpha-1} dt\Big| = B_2^1 + B_2^2.$$

By (2.33),

$$B_2^1 \le \int_{-x-\sqrt{x}}^{-x} d|\nu|(y) \le (x+\sqrt{x})^\delta \int_{-x-\sqrt{x}}^{-x} |y|^{-\delta} d|\nu|(y)\ .\ x^{\delta/2}$$

and

$$B_2^2 \le \frac{1}{\sqrt{x}} \int_{-\infty}^{-x-\sqrt{x}} d|\nu|(y) \int_0^x dt \simeq \sqrt{x}.$$

This gives the required estimate.

Let us estimate $k^-(x)$ for $|x| < 1$. By taking into account that k^- is odd we consider only the case $0 < x < 1$. We have for $\alpha \ne 0$:

$$2xk^-(x) = \frac{1}{\Gamma(\alpha+1)} \int_{-x}^{x} (x-y)^\alpha d\nu(y) - \frac{2}{\Gamma(\alpha+1)} \int_{-x}^{0} |y|^\alpha d\nu(y) +$$

$$+ \frac{1}{\Gamma(\alpha+1)} \int_{-\infty}^{-x} [(x-y)^\alpha + (-x-y)^\alpha - 2|y|^\alpha] d\nu(y) = J_1 + J_2,$$

where $|J_1|\ .\ \left| \int_{-x}^{x} (x-y)^\alpha d\nu(y) \right| + \left| \int_{-x}^{0} |y|^\alpha d\nu(y) \right| \simeq x^\delta,$ (cf. (A2.1)) and

$$|J_2|\ .\ \left| \int_{-\infty}^{-x} d\nu(y) \int_0^x (|y|+t)^{\alpha-1} dt \right| + \left| \int_{-\infty}^{-x} d\nu(y) \int_0^x (|y|-t)^{\alpha-1} dt \right| = B_1 + B_2.$$

(cf. (A2.2)). Proceeding as in Appendix 1 (see the estimate of F_2 in (A2.2)) we obtain what was required. If $\alpha = 0$, then

$$|2xk^-(x)| = |\mu(0,x) - \mu(-x,0)|\ .\ |x|^\delta. \qquad \Lambda$$

Appendix 3

Let us prove the second equality in (2.16) for $Re\,\alpha \ge 0$.

Step 1. Consider the case $0 \le Re\, \alpha < 1,\ \alpha \neq 0$. We have

$$\text{(A3.1)} \qquad \gamma_+ = \int_{-\infty}^{\infty} k^+(x)dx = \int_{|x|<1} k^+(x)dx = \int_{|x|>1} k^+(x)dx = I_1 + I_2.$$

The first integral has the form

$$I_1 = 2\int_0^1 k^+(x)dx = \frac{1}{\Gamma(\alpha+1)} \int_0^1 \frac{dx}{x}\Big(\int_{-x}^{x}(x-y)^\alpha d\nu(y) + \int_{-\infty}^{-x}((x-y)^\alpha - (x-y)^\alpha)d\nu(y)\Big)$$

$$= \frac{1}{\Gamma(\alpha+1)}\Big(\int_{-1}^{0} d\nu(y)\int_{|y|}^{1} \frac{(x+|y|)^\alpha}{x}dx + \int_0^1 d\nu(y)\int_y^1 \frac{(x-y)^\alpha}{x}dx +$$

$$+ \int_{-\infty}^{-1} d\nu(y)\int_0^1 \frac{(|y|+x)^\alpha - (|y|-x)^\alpha}{x}dx + \int_{-1}^{0} d\nu(y)\int_0^{|y|} \frac{(|y|+x)^\alpha - (|y|-x)^\alpha}{x}dx\Big)$$

$$= A_1 + A_2 + A_3 + A_4.$$

For the second integral in (A3.1) we have

$$I_2 = \int_{\infty}^{-1} (I_+^{1+\alpha}\nu)(x)\frac{dx}{x} + \int_1^{\infty} (I_+^{1+\alpha}\nu)(x)\frac{dx}{x} = I_2^1 + I_2^2, \qquad \text{where}$$

$$I_2^1 = \frac{1}{\Gamma(1+\alpha)}\int_{-\infty}^{-1}\frac{dx}{x}\int_{-\infty}^{x}(x-y)^\alpha d\nu(y) = -\frac{1}{\Gamma(1+\alpha)}\int_{-\infty}^{-1} d\nu(y)\int_1^{|y|}\frac{(|y|-x)^\alpha}{x}dx,$$

and (since $\nu(\mathrm{R}) = 0$)

$$I_2^2 = \int_1^\infty (I_+^{1+\alpha}\nu)(x)\frac{dx}{x} = \frac{1}{\Gamma(1+\alpha)}\int_1^\infty \Big[\int_{-\infty}^{-x}((x-y)^\alpha - x^\alpha)d\nu(y) - x^\alpha\int_x^\infty d\nu(y)\Big]\frac{dx}{x} =$$

$$= \frac{1}{\alpha\Gamma(1+\alpha)}\int_1^\infty (1-y)^\alpha d\nu(y) + \frac{1}{\Gamma(\alpha+1)}\int_{-\infty}^{1} d\nu(y)\int_1^\infty \frac{(x-y)^\alpha - x^\alpha}{x}dx +$$

$$+ \frac{1}{\Gamma(\alpha+1)}\int_1^\infty d\nu(y)\int_y^\infty \frac{(x-y)^\alpha - x^\alpha}{x}dx = B_1 + B_2 + B_3.$$

Our goal is to simplify the sum $I_1 + I_2 = A_1 + A_2 + A_3 + A_4 + I_2^1 + B_1 + B_2 + B_3$.

To do this we transform A_3 and B_2 as follows:

$$\frac{A_3}{\Gamma(1+\alpha)} = \int_{-\infty}^{-1} d\nu(y) \int_0^1 \frac{(|y|+x)^\alpha - (|y|-x)^\alpha}{x} dx = \int_{-\infty}^{-1} d\nu(y) \int_0^{|y|} \frac{(|y|+x)^\alpha - (|y|-x)^\alpha}{x} dx$$

$$- \int_{-\infty}^{-1} d\nu(y) \int_1^{|y|} \frac{(|y|+x)^\alpha}{x} dx + \int_{-\infty}^{-1} d\nu(y) \int_1^{|y|} \frac{(|y|-x)^\alpha}{x} dx$$

and

$$\Gamma(1+\alpha)B_2 = \int_{-\infty}^{1} d\nu(y) \int_1^{\infty} \frac{(x-y)^\alpha - x^\alpha}{x} dx = \int_{-\infty}^{0} d\nu(y) \int_{|y|}^{\infty} \frac{(x+|y|)^\alpha - x^\alpha}{x} dx +$$

$$+ \int_{-\infty}^{0} d\nu(y) \int_1^{|y|} \frac{(x+|y|)^\alpha - x^\alpha}{x} dx + \int_0^1 d\nu(y) \int_y^{\infty} \frac{(x-y)^\alpha - x^\alpha}{x} dx -$$

$$- \int_0^1 d\nu(y) \int_y^1 \frac{(x-y)^\alpha - x^\alpha}{x} dx = \int_{-\infty}^{0} d\nu(y) \int_{|y|}^{\infty} \frac{(x+|y|)^\alpha - x^\alpha}{x} dx +$$

$$+ \int_{-\infty}^{0} d\nu(y) \int_1^{|y|} \frac{(x+|y|)^\alpha}{x} dx - \frac{1}{\alpha} \int_{-\infty}^{0} (|y|^\alpha - 1) d\nu(y) +$$

$$+ \int_0^1 d\nu(y) \int_y^{\infty} \frac{(x-y)^\alpha - x^\alpha}{x} dx - \int_0^1 d\nu(y) \int_y^1 \frac{(x-y)^\alpha}{x} dx + \frac{1}{\alpha} \int_0^1 (1-y^\alpha) d\nu(y).$$

By taking into account the last equalities we have

$$\Gamma(1+\alpha)\gamma_+ = -\frac{1}{\alpha} \int_{-\infty}^{\infty} |y|^\alpha d\nu(y) + \int_0^{\infty} d\nu(y) \int_y^{\infty} \frac{(x-y)^\alpha - x^\alpha}{x} dx +$$

$$+ \int_{-\infty}^{0} d\nu(y) \int_{|y|}^{\infty} \frac{(x+|y|)^\alpha - x^\alpha}{x} dx + \int_{\infty}^{0} d\nu(y) \int_0^{|y|} \frac{(|y|+x)^\alpha - (|y|-x)^\alpha}{x} dx =$$

(A3.2) $$= K_1 \int_0^{\infty} y^\alpha d\nu(y) + K_2 \int_{-\infty}^{0} |y|^\alpha d\nu(y), \qquad \text{where}$$

(A3.3) $$K_1 = -\frac{1}{\alpha} + \int_1^{\infty} \frac{(\xi-1)^\alpha - \xi^\alpha}{\xi} d\xi = -\frac{1}{\alpha} + \int_0^1 \frac{(1-\eta)^\alpha - 1}{\eta^{1+\alpha}} d\eta = \Gamma(1+\alpha)\Gamma(-\alpha)$$

(use integration by parts),

$$K_2 = -\frac{1}{\alpha} + \int_1^\infty \frac{(\xi+1)^\alpha - \xi^\alpha}{\xi} d\xi + \int_0^1 \frac{(\xi+1)^\alpha - (1-\xi)^\alpha}{\xi} d\xi = -\frac{1}{\alpha} + \Lambda_1 + \Lambda_2.$$

Let us evaluate K_2. For Λ_1 by putting $\rho = \int_1^\infty \frac{(\xi+1)^{\alpha-1}}{\xi} d\xi$ we have

$$\Lambda_1 = \rho + \Big(\int_0^\infty - \int_0^1 \Big) \Big[(\xi+1)^{\alpha-1} - \xi^{\alpha-1} \Big] d\xi = \rho + \int_0^\infty \frac{(\eta+1)^{\alpha-1} - 1}{\eta^{1+\alpha}} d\eta - \frac{2^\alpha - 1}{\alpha} + \frac{1}{\alpha} =$$

$$= \rho - \frac{1-\alpha}{\alpha} \int_0^\infty \frac{(\eta+1)^{\alpha-2}}{\eta^\alpha} d\eta + \frac{2-2^\alpha}{\alpha} = \rho + \frac{1-2^\alpha}{\alpha}.$$

For Λ_2 in the case $0 < Re\,\alpha < 1$ we obtain

$$\Lambda_2 = \lim_{\varepsilon \to 0} \Big[\int_\varepsilon^1 \frac{(\xi+1)^{\alpha-1}}{\xi} d\xi + \int_\varepsilon^1 (\xi+1)^{\alpha-1} d\xi + \int_{-1}^{-\varepsilon} \frac{(\xi+1)^{\alpha-1}}{\xi} d\xi + \int_{-1}^{-\varepsilon} (\xi+1)^{\alpha-1} d\xi \Big] =$$

$$= p.v. \int_{-1}^1 \frac{(\xi+1)^{\alpha-1}}{\xi} d\xi + \frac{2^\alpha}{\alpha}.$$

By formula 3.22 (2) (Gradshteyn and Ryzhik, 1980) this yields

$$K_2 = p.v. \int_{-1}^\infty \frac{(\xi+1)^{\alpha-1}}{\xi} d\xi = -\pi ctg(\alpha\pi) = \Gamma(1+\alpha)\Gamma(-\alpha)\cos(\alpha\pi).$$

The equality

$$K_2 = \Gamma(1+\alpha)\Gamma(-\alpha)\cos(\alpha\pi) \tag{A3.4}$$

which was obtained above for $0 < Re\,\alpha < 1$, can be easily extended to $Re\,\alpha = 0$ $(\alpha \neq 0)$ by passing to the limit as $Re\,\alpha \to +0$, $Im\,\alpha = const$. From (A3.2)-(A3.4) one has (2.16).

Step 2. Consider the case $\ell \leq Re\,\alpha < \ell + 1$, $\ell \in \mathrm{N}, \alpha \neq \ell$. Put $\alpha_0 = \alpha - \ell$, $u = I_+^\ell \nu$. Then $0 \leq Re\,\alpha_0 < 1$, and $I_+^\alpha \nu = I_+^{\alpha_0} u$. In order to use Step 1, we first check the following conditions:

a) $\int_{-\infty}^\infty u(x) dx = 0$, b) $\int_{|x|>1} |x|^{\beta_0} \, |u(x)| dx < \infty$ for some $\beta_0 > Re\,\alpha_0$,

c) $\int_{|x|<1} |x|^{-\delta_0} |u(x)| dx < \infty$ for some $\delta_0 \in (0,1)$.

(the condition c) is needed only in the case $Re\,\alpha = \ell$). By Lemma 2.1, $u = I_+^\ell \nu \in L^1$ and $\hat{u}(\xi) = (-i\xi)^{-\ell}\hat{\nu}(\xi) = (I_+^{\ell+\xi}\nu)\hat{(}\xi)(i\xi)^\varepsilon$, $\varepsilon \in (0,1)$. Since $I_+^{\ell+\varepsilon}\nu \in L^1$ (see Lemma 2.1), then $\hat{u}(0) = 0$ that gives a).

Let us check b). For $\beta_0 = \beta - \ell > 0$ we have

$$\int_{-\infty}^{-1} |x|^{\beta_0}|u(x)|dx \,.\, \int_{-\infty}^{-1} |y|^\beta d|\nu|(y) \int_{1/|y|}^{1} (1-t)^{\ell-1}t^{\beta-\ell}dt \le c \int_{-\infty}^{-1} |y|^\beta d|\nu|(y) < \infty.$$

Since (by the first condition in 2.9)), $u \equiv I_+^\ell \nu = (-1)^\ell I_-^\ell \nu$, then

$$\int_1^\infty x^{\beta_0}|u(x)|dx \,.\, \int_1^\infty x^{\beta_0}dx \int_x^\infty (y-x)^{\ell-1}d|\nu|(y) < \infty \qquad \text{as above.}$$

Let us check c). Note, that by the first condition in (2.9),

$$I_+^\ell \nu = (-1)^\ell I_-^\ell \nu. \tag{A3.5}$$

This gives

$$\int_0^1 x^{-\delta_0}|u(x)|dx \,.\, \int_1^\infty y^{\ell-\delta_0}d\nu(y) \int_0^{1/y} (1-t)^{\ell-1}t^{-\delta_0}dt + \int_0^1 y^{\ell-\delta_0}d\nu(y) \int_0^1 (1-t)^{\ell-1}t^{-\delta_0}dt < \infty$$

by (2.10′). Similarly for $u = I_+^\ell \nu$,

$$\int_{-1}^0 |x|^{-\delta}|u(x)|dx \,.$$

$$.\int_{-\infty}^{-1} |y|^{\ell-\delta_0}d|\nu|(y) \int_0^{1/|y|} (1-t)^{\ell-1}t^{-\delta_0}dt + \int_{-1}^0 |y|^{\ell-\delta_0}d|\nu|(y) \int_0^1 (1-t)^{\ell-1}t^{-\delta_0}dt < \infty.$$

Now in accordance with Step 1 by changing the order of integration we have:

$$\int_{-\infty}^\infty k^+(x)dx = \Gamma(-\alpha_0)\Big(\int_0^\infty x^{\alpha_0}u(x)dx + \cos(\alpha_0\pi)\int_{-\infty}^0 |x|^{\alpha_0}u(x)dx\Big) =$$

$$= (-1)^\ell \frac{\Gamma(-\alpha_0)}{\Gamma(\ell)} B(\alpha_0+1,\ell)\Big(\int_0^\infty x^\alpha d\nu(x) + \cos(\alpha_0\pi)(-1)^\ell \int_{-\infty}^0 |x|^\alpha d\nu(x)\Big) =$$

$$= \Gamma(-\alpha)\Big(\int_0^\infty x^\alpha d\nu(x) + \cos(\alpha\pi)\int_{-\infty}^0 |x|^\alpha d\nu(x)\Big).$$

Step 3. Let $\alpha = \ell$ be a nonnegative integer. Note that

$$\text{(A3.6)} \qquad \gamma_+ = \int_{-\infty}^{\infty} k^+(x)dx = \int_{-\infty}^{\infty} \log\frac{1}{|x|}(I_+^\ell \nu)(x)dx.$$

Indeed, since $k^+(x)$ is even, then

$$\gamma_+ = 2\int_0^1 k^+(x)dx + 2\int_1^\infty k^+(x)dx = U_1 + U_2,$$

where (cf. (2.8))

$$U_1 = \int_0^1 (I_+^\ell \nu)(y)dy \int_y^1 \frac{dx}{x} + \int_{-1}^0 (I_+^\ell \nu)(y)dy \int_{-y}^1 \frac{dx}{x} = \int_{-1}^1 \log\frac{1}{|y|}(I_+^\ell \nu)(y)dy$$

and (use the equality $\int_{\infty}^{\infty}(I_+^\ell \nu)(x)dx = 0$, (see step 2))

$$U_2 = -\int_1^\infty \frac{dx}{x}\int_x^\infty (I_+^\ell \nu)(y)dy - \int_1^\infty \frac{dx}{x}\int_{-\infty}^{-x} (I_+^\ell \nu)(y)dy = \int_{|y|>1} \log\frac{1}{|y|}(I_+^\ell \nu)(y)dy.$$

From (A3.6), (A3.5) we get

$$\gamma_+ = \frac{1}{\Gamma(\ell)}\int_{-\infty}^{0} |y|^\ell d\nu(y)\int_0^1 (1-t)^{\ell-1}\log\frac{1}{t|y|}dt + \frac{(-1)^\ell}{\Gamma(\ell)}\int_0^\infty y^\ell d\nu(y)\int_0^1 (1-t)^{\ell-1}\log\frac{1}{ty}dt =$$

$$= \frac{(-1)^\ell}{\ell!}\int_{-\infty}^{\infty} y^\ell \log\frac{1}{|y|}d\nu(y).$$

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