Real Analysis, Math 821.

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Assignment I.

Notation. Let X be a set. By $P(X)$ we will denote the collection of all subsets of X.

1. **Problem 1.** Let X consist of n elements. Prove that $P(X)$ consists of 2^n elements.

   Definition 1. A nonempty subcollection $R \subset P(X)$ is called **a ring of subsets of** X, or just a ring, provided R is closed under the union, intersection, and difference. In other words, for any $A, B \subset R$, $A \cap B \subset R$, $A \cup B \subset R$, $A \setminus B \subset R$. Observe that this implies also $A \triangle B := (A \setminus B) \cup (B \setminus A) \subset R$ (the last operation is called the symmetric difference).

   Examples. For any X, $P(X)$ is a ring. A set of all bounded subsets of the real line is a ring. A collection $\{A, \varnothing\}$, (where $\varnothing$ is an empty set), is a ring.

2. **Problem 2.** Let a subcollection $R \subset P(X)$ be such that for any $A, B \subset R$, $A \cap B \subset R$, and $A \cup B \subset R$. Prove that R is not (in general) a ring. On the other hand, if for any $A, B \subset R$, $A \cup B \subset R$, and $A \setminus B \subset R$, then R is a ring.

   Definition 2. A nonempty subcollection $\aleph \subset P(X)$ closed under intersection is called a **subring** , if $\varnothing \subset \aleph$, and if $A, A_1 \subset \aleph$, $A_1 \subset A$, then $A = \cup_{k=1}^{n} A_k$, where all A_k are disjoint.

3. **Problem 3.** Prove that the set of all segments (open, closed, semi-open) on the real line is a subring, but not a ring.

4. **Problem 4.** Let $X := \{a, b, c\}$, be a set consisting of three elements.

   a) Write out $P(X)$.

   b) Give an example of a subring, consisting of elements of $P(X)$, which is not a ring.

   c) Discribe all subrings, which can be obtained from the elements of X.

   d) Discribe all rings, which can be obtained from the elements of X.

   Definition 3. Let $A \subset X$. A function

$$1_A(x) := \begin{cases} 1 & \text{for } x \in A \\ 0 & \text{for } x \notin A, \end{cases}$$

   defined on X, is a called a **characteristic function of a set** A.

5. **Problem 5.** Prove that for $A, B \subset X$,

   a) $1_{A \cap B}(x) = 1_A(x) 1_B(x)$,

   b) $1_{A \cup B}(x) = 1_A(x) + 1_B(x) - 1_A(x) 1_B(x)$,

   c) $1_{A \triangle B}(x) = 1_A(x) + 1_B(x)$,

   d) $1_{A \setminus B}(x) = 1_A(x) - 1_A(x) 1_B(x)$.

6. **Problem 6.** Let R be a collection of subsets of X, and let $\tilde{R}$ be a collection of characteristic functions of sets belonging to R. Prove that R is a ring of subsets of X if and only if $\tilde{R}$ is a usual ring of functions (in other words, if $1_A(x), 1_B(x) \in \tilde{R}$, then $1_A(x)1_B(x) \in \tilde{R}$, $1_A(x) + 1_B(x) \in \tilde{R}$).

7. **Problem 7*.** Assume that you have 10 sets. How many **disjoint nonempty** sets you can obtain by taking union, intersection, difference, and symmetric difference of the given sets? You may apply these operations as many times as you want.

   Hint: Consider the sets $A_1, A_2, ..., A_{10}$ of sequences of length 10 consisting only of $1, 0$, where all sequences belonging to A_i have 1 in the $i-$th place, $i = 1, ..., 10$.