

Real Analysis, Math 821.

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Assignment I.

Notation. Let X be a set. By $P(X)$ we will denote the collection of all subsets of X .

1. **Problem 1.** Let X consist of n elements. Prove that $P(X)$ consists of 2^n elements.

Definition 1. A nonempty subcollection $R \subset P(X)$ is called a **ring of subsets of X** , or just a ring, provided R is closed under the union, intersection, and difference. In other words, for any $A, B \in R$, $A \cap B \in R$, $A \cup B \in R$, $A \setminus B \in R$. Observe that this implies also $A \Delta B := (A \setminus B) \cup (B \setminus A) \in R$ (the last operation is called the symmetric difference).

Examples. For any X , $P(X)$ is a ring. A set of all bounded subsets of the real line is a ring. A collection $\{A, \emptyset\}$, (where \emptyset is an empty set), is a ring.

2. **Problem 2.** Let a subcollection $R \subset P(X)$ be such that for any $A, B \in R$, $A \cap B \in R$, and $A \cup B \in R$. Prove that R is not (in general) a ring. On the other hand, if for any $A, B \in R$, $A \cup B \in R$, and $A \setminus B \in R$, then R is a ring.

Definition 2. A nonempty subcollection $\mathfrak{N} \subset P(X)$ closed under intersection is called a **subring**, if $\emptyset \in \mathfrak{N}$, and if $A, A_1 \in \mathfrak{N}$, $A_1 \subset A$, then $A = \bigcup_{k=1}^n A_k$, where all A_k are disjoint.

3. **Problem 3.** Prove that the set of all segments (open, closed, semi-open) on the real line is a subring, but not a ring.

4. **Problem 4.** Let $X := \{a, b, c\}$, be a set consisting of three elements.

a) Write out $P(X)$.

b) Give an example of a subring, consisting of elements of $P(X)$, which is not a ring.

c) Describe all subrings, which can be obtained from the elements of X .

d) Describe all rings, which can be obtained from the elements of X .

Definition 3. Let $A \subset X$. A function

$$1_A(x) := \begin{cases} 1 & \text{for } x \in A \\ 0 & \text{for } x \notin A, \end{cases}$$

defined on X , is called a **characteristic function of a set A** .

5. **Problem 5.** Prove that for $A, B \subset X$,

a) $1_{A \cap B}(x) = 1_A(x)1_B(x)$,

b) $1_{A \cup B}(x) = 1_A(x) + 1_B(x) - 1_A(x)1_B(x)$,

c) $1_{A \Delta B}(x) = 1_A(x) + 1_B(x)$,

d) $1_{A \setminus B}(x) = 1_A(x) - 1_A(x)1_B(x)$.

6. **Problem 6.** Let R be a collection of subsets of X , and let \tilde{R} be a collection of characteristic functions of sets belonging to R . Prove that R is a ring of subsets of X if and only if \tilde{R} is a usual ring of functions (in other words, if $1_A(x), 1_B(x) \in \tilde{R}$, then $1_A(x)1_B(x) \in \tilde{R}$, $1_A(x) + 1_B(x) \in \tilde{R}$).
7. **Problem 7*.** Assume that you have 10 sets. How many **disjoint nonempty** sets you can obtain by taking union, intersection, difference, and symmetric difference of the given sets? You may apply these operations as many times as you want.
- Hint:** Consider the sets A_1, A_2, \dots, A_{10} of sequences of length 10 consisting only of 1, 0, where all sequences belonging to A_i have 1 in the i -th place, $i = 1, \dots, 10$.