Real Analysis, Math 821.

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Assignment X.

1. **Problem 1.**

   a) Is it possible to construct $f(x)$, $x \in [0,1]$, such that $f'(x) = D(x)$? Here $D(x) = 1$ for $x \in [0,1] \cap \mathbf{Q}$, and $D(x) = 0$ for $x \in [0,1] \cap \mathbf{R} \setminus Q$.

   Hint: Prove the **Darboux Theorem**: *If $f(x)$ is differentiable on $[0,1]$, then $\forall C \in [f'(0), f'(1)]$ there exists $x \in [0,1]$ such that $f'(x) = C$.*

   b) Construct a function $f(x)$ on $[0,1]$ such that $f'(x)$ exists at every $x \in [0,1]$, (and bounded), but $f'(x)$ is not continuous for every $x \in F$, where $F \subset [0,1]$, and $m(F) > 0$.

   Hint: Let $F \subset [0,1]$, $m(F) > 0$ be closed, nowhere dense, and such that $\inf F = 0$, $\sup F = 1$. Define

   $$f(x) = (x - a_n)^2 (x - b_n)^2 \sin \frac{1}{(b_n - a_n)(x - a_n)(x - b_n)}, \qquad x \in (a_n, b_n),$$

   where $[0,1] \setminus F = \cup_{i=1}^{\infty} (a_i, b_i)$, and $f(x) = 0$ otherwise.

   c) Construct a **continuous** function f on $\mathbf{R}$ which is not differentiable at any point.

   Hint: Put $\phi(x) = x$, $x \in [0,1]$, and $\phi(x) = 2 - x$, $x \in [1,2]$. Define $\phi_0(x) := \phi(x)$, $x \in [0,2]$, and $\phi_0(x+2) = \phi_0(x)$. Then, define $f(x) := \sum\limits_{n=0}^{\infty} (3/4)^n \phi_0(4^n x)$.

   a) Fix any $x \in \mathbf{R}$ and any $m \in \mathbf{N}$. Observe that there exists $k \in \mathbf{Z}$ such that $k \le 4^m x \le k+1$, and put $\alpha_m := 4^{-m} k$, $\beta_m := 4^{-m}(k+1)$. Prove that $|\phi_0(4^n \beta_m) - \phi_0(4^n \alpha_m)| = 0$, for $n > m$, and $|\phi_0(4^n \beta_m) - \phi_0(4^n \alpha_m)| = 4^{n-m}$ for $n \le m$.

   b) Conclude that $|f(\beta_m) - f(\alpha_m)| \ge 1/2(3/4)^m$, and show that f is not differentiable at x.

2. **Problem 2.** a) Let $f(0) = 0$, $f(1) = 5$, $f(x) = 1 - x$, for $x \in (0,1)$. Use definition to find the total variation of $f(x)$ on $[0,1]$.

   b) Write out $f(x) = \cos^2 x$ on $[0, \pi]$ as a difference of two increasing functions.

   c) Let $f(x) = x^2$, $x \in [0,1)$, $f(x) = x + 3$, $x \in (1,2]$, $f(1) = 5$. Check that $V_0^2(f) = V_0^1(f) + V_1^2(f)$. Write out $f(x)$ as a difference of two increasing functions.

3. **Problem 3.** a) Let $f : V_0^1(|f|) < \infty$. Is it true that $V_0^1(f) < \infty$?

   b) Let f be **continuous** on $[0,1]$, and such that $V_0^1(|f|) < \infty$. Prove that $V_0^1(f) < \infty$.

   Hint: Use the mean-value theorem.

4. **Problem 4.** a) Construct a continuous $f(x)$ on $[a,b]$ such that $V_a^b(f) < \infty$, but $f(x)$ "is not Holder" for any $\alpha > 0$, (f is said to satisfy the Holder condition for some $\alpha > 0$ on $[a,b]$, if there exists a constant $K > 0$ such that

$$\forall x, y \in [a,b], \qquad |f(x) - f(y)| \le K|x-y|^{\alpha}).$$

Hint: Take $[a,b] = [0, 1/2]$, and $f(x) = -1/\log x$, $x \in (0, 1/2]$, $f(0) = 0$.

b)* Construct an example of a continuous $f(x)$ on $[a,b]$, such that $V_a^b(f) = \infty$, but f "is Holder" of the given $0 < \alpha < 1$.

Hint: Let $(a_i)_{i=1}^{\infty}$ be such that $a_i > a_{i+1} > 0$ and $\sum\limits_{i=1}^{\infty} a_i = A$. Put $f(x) = 0\ \forall x = a_1, a_1 + a_2, a_1 + a_2 + a_3, ...,$; $f(x) = 1/n$ at the point $a_1 + a_2 + a_+ ... + a_{n-1} + a_n/2$, $n = 1, 2, 3, ...$; $f(1) = 0$, and make f to be linear on any segment of the type $[\sum\limits_{i=1}^{n-1} a_i, \sum\limits_{i=1}^{n-1} a_i + a_n/2]$, $[\sum\limits_{i=1}^{n-1} a_i + a_n/2, \sum\limits_{i=1}^{n} a_i]$, and on the segments $[0, a_1/2]$, $[a_1/2, a_1]$.

To show that f "is Holder" of the given $0 < \alpha < 1$, take $a_n := n^{-1/\alpha}$, and consider two cases, 1) points $M_1(x_1, y_1)$, $M_2(x_2, y_2)$ belong to "the same" part of the graph of $f(x)$, 2) points $M_1(x_1, y_1)$, $M_2(x_2, y_2)$ do not belong to "the same" part of the graph of $f(x)$.

5. **Problem 5.** a) Let $(f_n(x))_{n=1}^{\infty}$ be a sequence of functions having bounded variation on $[a,b]$. Assume also that $\sum\limits_{n=1}^{\infty} V_a^b(f_n) < \infty$, and $f_n(a) = 0$, $\forall n \in \mathbf{N}$. Prove that the series $\sum\limits_{n=1}^{\infty} f_n(x)$ is convergent $\forall x \in [a,b]$, and $V_a^b(\sum\limits_{n=1}^{\infty} f_n) \le \sum\limits_{n=1}^{\infty} V_a^b(f_n)$.

b) Let $(f_n(x))_{n=1}^{\infty}$ be a sequence of continuous functions having bounded variation on $[a,b]$. Assume also that the series $\sum\limits_{n=1}^{\infty} f_n(x)$ converges uniformly on $[a,b]$. Is it true that $V_a^b(\sum\limits_{n=1}^{\infty} f_n) < \infty$?

Hint: Consider $(f_n(x))_{n=1}^{\infty}$ on $[0,1]$, $f_n(x) := \sin(n\pi(x(n+1)-1))/n$ on $[1/(n+1), 1/n]$, $f_n(x) := 0$ on $[0,1] \setminus [1/(n+1), 1/n]$. You may also use an example from Problem 1, c).

c) Construct a function f, which is of bounded variation on any finite segment (and hence is a difference of two monotonic functions), but, nevertheless, is not monotonic on any segment.

Hint: Let $\phi_0(x) = |x|$ for $x \in [-1/2, 1/2]$, $\phi_0(x+1) = \phi_0(x)$, and let $\phi_n(x) := \min(\phi_0(x), 8^{-n})$. Consider $f(x) := \sum\limits_{n=0}^{\infty} 2^{-n}\phi_n(8^n x)$.