Real Analysis, Math 821.

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Assignment XI.

1. **Problem 1.** a) Let $E := \cup_{k=1}^{\infty} E_k \subset \mathbf{R}$, $E_k := [2^{-2k-1}, 2^{-2k}] \cup [-2^{-2k}, -2^{-2k-1}]$. Show that at 0, E has upper density 2/3 and lower density 1/3. In other words, show that

$$\overline{\lim_{\delta\to 0}} \frac{m(E \cap (-\delta,\delta))}{2\delta} = \frac{2}{3}, \qquad \underline{\lim_{\delta\to 0}} \frac{m(E \cap (-\delta,\delta))}{2\delta} = \frac{1}{3}.$$

   b) Construct a set $E \subset \mathbf{R}$ which has density 1/2 at the origin.

   c) Suppose that E is a measurable set of real numbers with arbitrary small periods, i.e., there exists a sequence $(p_i)_{i=1}^{\infty}$, $p_i > 0$, $p_i \to 0$ as $i \to \infty$, so that $E + p_i = E$ $\forall i$. Prove that either $m(E) = 0$ or $m(\mathbf{R} \setminus E) = 0$.

   Hint: Pick $\alpha \in \mathbf{R}$, and put $f(x) := m(E \cap [\alpha, x])$ for $x > \alpha$. Show that

$$f(x+p_i) - f(x-p_i) = f(y+p_i) - f(y-p_i), \qquad \alpha + p_i < x < y.$$

   What does this imply about $f'(x)$ if $m(E) > 0$?

   d) Suppose that f is Lebesgue measurable on $\mathbf{R}$ with periods s, t, $s/t \in \mathbf{R}\setminus\mathbf{Q}$ ($\forall x \in \mathbf{R}$, $f(x+t) = f(x), f(x+s) = f(x)$). Prove that $f(x) = const$ almost everywhere, but f need not to be a constant.

   Hint: Apply c) to the set $\{x \in \mathbf{R} : \ f(x) > \lambda\}$, where $\lambda > 0$.

2. **Problem 2.** In this exercise f is a continuous function on $[a, b]$, $F, E \subset [a, b]$ are measurable.

   a) Let F be closed. Prove that $f(F)$ is closed.

   b) Let F be F_σ, then $f(F)$ is F_σ.

   Definition (Lusin). We say that f has N-property, if $m(f(E)) = 0$ $\forall E : \ m(E) = 0$.

   c) Prove that absolutely continuous function has N-property.

   d) Show that $f(E)$ is measurable $\forall E$ measurable **iff** f has N-property.

   Hint. To prove part **if**, observe that f does not have N-property, implies the existence of E, such that $m(E) = 0$, and $\mu^*(f(E)) > 0$. Then there exists $B \subset f(E)$ which is not measurable (why?). Consider $A := f^{-1}(B) \subset E$. Show that A is measurable, and get a contradiction.

   e) Prove that an absolutely continuous function maps measurable sets onto measurable sets.

   f) Construct a **continuous** one-to-one mapping of $[0, 1]$ onto $[0, 2]$ which does not have N-property.

   Hint: Consider $f(x) = c(x) + x$, where $c(x)$ is the Cantor ladder, defined in Problem 3, Assignment 7. What is the measure of the image of the Cantor set?

3. **Problem 3.** This exercise helps to understand "the change of variables" in the Lebesgue integral.

   Let $(a, b) \subset \mathbf{R}$ be bounded.

   a) Choose intervals $W_n \subset (a, b)$ in such away that $\cup_n W_n$ is dense in (a, b), and the set $K := (a, b) \setminus \cup_n W_n$ has positive measure.

   b) Choose continuous functions ϕ_n on W_n so that $\phi_n(x) = 0$ outside W_n, and $0 < \phi_n(x) < 2^{-n}$ in W_n.

   Put

$$\phi(x) := \sum_{n=1}^{\infty} \phi_n(x), \qquad T(x) := \int_a^x \phi(t)dt, \qquad x \in (a, b).$$

   c) Prove that T is one-to-one, and differentiable at every point.

   d) Prove that T' is continuous, $T'(x) = 0$, $\forall x \in K$, $m(T(K)) = 0$.

   e) Prove that if E is a nonmeasurable subset of K (why it exists?), then $1_{T(E)}(x)$ is measurable, but $1_{T(E)}(Tx)$ is not.

   Observe that ϕ_n can be chosen such that T is *infinitely differentiable homeomorphism* of (a, b) onto some segment in $\mathbf{R}$.

4. **Problem 4.** This exercise is a part of the proof of "the change of variables" in the Lebesgue integral.

   Let ϕ be increasing and absolutely continuous function on $[p, q]$, $x = \phi(t)$, $a = \phi(p)$, $b = \phi(q)$.

   a) Let E_t me a measurable subset of $[p, q]$, $E_x := \phi(E_t)$. Prove that

$$m(E_x) = \int_{E_t} \phi'(t)dt$$

   Hint. Prove this at first for $E_t = [\alpha, \beta]$. Then for (α, β), open and closed E_t. In the general case, find closed F_x, and open G_x such that $\forall \epsilon > 0$,

$$F_x \subset E_x \subset G_x \subset (a, b), \qquad m(F_x) > m(E_x) - \epsilon, \qquad m(G_x) < m(E_x) + \epsilon.$$

   Consider $F_t := \phi^{-1}(F_x)$, $G_t := \phi^{-1}(G_x)$, and observe that

$$m(F_x) \leq \int_{E_t} \phi'(t)dt \leq m(G_x).$$

   b) Let all notation be as in a), and let $e_x \subset [a, b]$, $m(e_x) = 0$, $e_t = \phi^{-1}(e_x)$. Prove that

$$m(e_t^*) = 0, \qquad e_t^* := \{t \in e_t : \phi'(t) = 0 \text{ is false}\}.$$

Hint: Observe that

$$e_x \subset G_x^n \subset G_x^{n-1} \subset ... \subset G_x^2 \subset G_x^1 \subset (a,b), \qquad m(G_x^n) \to 0.$$

Define

$$H_x := \cap_{n=1}^{\infty} G_x^n, \qquad H_t := \phi^{-1}(H_x), \qquad G_t^n := \phi^{-1}(G_x^n),$$

and prove that H_t is measurable, and $m(H_x) = 0$. Show that $\int\limits_{H_t} \phi'(t)dt = 0$.