# Real Analysis, Math 821.

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# Assignment XII.

1. **Problem 1.** a) Let $\mu$ be a complex measure on a $\sigma$-algebra $\Sigma$, and let $E \in \Sigma$. Define $\lambda(E) := \sup \sum |\mu(E_i)|$, the supremum being taken over all **finite** partitions $\{E_i\}$ of $E$. Does it follow that $\lambda = |\mu|$?

   b) Define a measure $\lambda$ on Lebesgue measurable subsets of $\mathbf{R}^2$ as follows

   $$\lambda(A) := \int\limits_{A \cap \mathbf{R}} f(x, 0) dx, \qquad A \subset \mathbf{R}^2,$$

   Here $f(x, 0)$ is a continuous function having a compact support on $\mathbf{R} = \{(x, y) : \ y = 0\}$.

   Prove that $\lambda$ is *absolutely continuous* with respect to the Lebesgue measure on $\mathbf{R}$, but is *singular* with respect to the Lebesgue measure on $\mathbf{R}^2$.

2. **Problem 2.**

   **Definition**. We say that two measures $\lambda_1, \lambda_2$, defined on a $\sigma$-algebra $\Sigma$ are **mutually singular**, ($\lambda_1 \perp \lambda_2$), if there exists a pair of **disjoint** sets $A, B$ such that $\lambda_1$ is concentrated on $A$, and $\lambda_2$ is concentrated on $B$.

   Suppose that $\mu, \lambda, \lambda_1, \lambda_2$ are measures on a $\sigma$-algebra $\Sigma$, and $\mu$ is positive. Prove the following chain of stements.

   a) If $\lambda$ is concentrated on $A$, so is $|\lambda|$.

   b) If $\lambda_1 \perp \lambda_2$, then $|\lambda_1| \perp |\lambda_2|$.

   c) If $\lambda_1 \perp \mu$, and $\lambda_2 \perp \mu$, then $(\lambda_1 + \lambda_2) \perp \mu$.

   d) If $\lambda_1 \ll \mu$ and $\lambda_2 \ll \mu$, then $(\lambda_1 + \lambda_2) \ll \mu$.

   e) If $\lambda \ll \mu$, then $|\lambda| \ll \mu$.

   f) If $\lambda_1 \ll \mu$ and $\lambda_2 \perp \mu$, then $\lambda_1 \perp \lambda_2$.

   g) If $\lambda \ll \mu$ and $\lambda \perp \mu$, then $\lambda = 0$.

3. **Problem 3.** Suppose $\mu$ and $\lambda$ are measures on a $\sigma$-algebra $\Sigma$, $\mu$ is positive, $\lambda$ is complex. Prove that the following two conditions are equivalent:

   $\aleph$) $\lambda \ll \mu$.

   $\beth$) For every $\epsilon > 0$ there exists a $\delta > 0$ such that $|\lambda(E)| < \epsilon$ for all $E \in \Sigma$ with $\mu(E) < \delta$.

   **Hint**. To show that $\aleph$) implies $\beth$), assume that $\beth$) is false. Then there exists $\epsilon > 0$ and $E_n \in \Sigma$ such that $\mu(E_n) < 2^{-n}$, but $|\lambda(E_n)| \geq \epsilon$. Put $A_n := \cup_{i=n}^{\infty} E_i$, $A := \cup_{n=1}^{\infty} A_n$. Prove that $\mu(A) = 0$, but $|\lambda|(A) > 0$. Use Problem 2, e), to get a contradiction.

4. **Problem 4.** Suppose $\mu$ is a positive measure on $\Sigma$, $g \in L(X,\mu)$, and $\lambda(E) = \int\limits_E f d\mu$, $E \in \Sigma$. Prove that $|\lambda|(E) = \int\limits_E |f| d\mu$.

   **Hint**. Observe that $|\lambda|(E) \leq \int\limits_E |f| d\mu$. To show the opposite inequality, construct a sequence $(g_n(x))_{n=1}^{\infty}$ of measurable simple functions such that $|g_n(x)| = 1$, and $\lim\limits_{n\to\infty} g_n(x)f(x) = |f(x)|$. Check that

$$|\int\limits_A g_n f d\mu| = |\sum_j a_{n,j} \int\limits_{A\cap A_{n,j}} f d\mu| = |\sum_j a_{n,j}\lambda(A \cap A_{n,j})| \leq \sum_j |\lambda(A \cap A_{n,j})| \leq |\lambda|(A),$$

   where $a_{n,j}$ are the values of $g_n$, attained on the sets $A_{n,j}$.

5. **Problem 5.** Let $(r_n)_{n=1}^{\infty}$ be an enumeration of the rational numbers, and for each positive integer $n$, let $f_n : \mathbf{R} \to \mathbf{R}$ be defined as $f_n(x) = 2^{n+1}$, $x \in [r_n - 2^{-n}, r_n + 2^{-n}]$, and zero otherwise. Define a measure $\lambda$ on Borel subsets of $\mathbf{R}$ by

$$\lambda(A) := \int\limits_A f(x)dx, \qquad f(x) := \sum_{n=1}^{\infty} f_n(x).$$

   a) Show that $f(x)$ is finite almost everywhere with respect to $m$, the Lebesgue measure on $\mathbf{R}$.

   **Hint**. Define $A_k := \{x \in \mathbf{R} : f(x) \geq 2^k\}$, $k$ is a nonnegative integer. To show that $m(\cap_{k=1}^{\infty} A_k) = 0$ observe that $\sum\limits_{k=1}^{\infty} m(A_k) < \infty$.

   b) Show that $\lambda$ is $\sigma$-finite. In other words find a partition of $\mathbf{R}$ into disjoint sets $B_k$, such that $\mathbf{R} = \cup_{k=1}^{\infty} B_k$, and $\lambda(B_k) < \infty$.

   **Hint**. One can take

$$B_1 := A_1^c, \qquad B_k := A_k^c \setminus (\cup_{l=1}^{k-1} A_l^c).$$

   c) Show that $\lambda \ll m$.

   d) Show that each non-empty open subset of $\mathbf{R}$ has infinite measure under $\lambda$.

   **Hint**. Observe that $f(x) \geq \sum\limits_{k=1}^{\infty} f_{n_k}(x)$, where a subsequence $n_k$ is chosen such that segments $[r_{n_k} - 2^{-n_k}, r_{n_k} + 2^{-n_k}]$, $[r_{n_l} - 2^{-n_l}, r_{n_l} + 2^{-n_l}]$ are disjoint (for $k \neq l$) and belong to the given open interval. Use $\int\limits_{\mathbf{R}} f_n(x)dx = 1$.