Real Analysis, Math 821.

Instructor: Dmitry Ryabogin

Assignment XIII.

1. **Problem 1.** a) Compute the Hardy-Littlewood maximal function $M1_{[0,1]}(x)$, of the characteristic function of segment $[0,1]$.

   b) Let $f(x) = x^{-1}(\log x)^{-2}$ if $x \in (0,1/2)$, and zero on the rest of $\mathbf{R}$. Prove that f is integrable. On the other hand, show that $Mf(x) \geq |2x\log(2x)|^{-1}$, if $x \in (0,1/4)$, so that $\int\limits_0^1 Mf(x)dx = \infty$.

   c) Assume that both f and Mf are integrable on $\mathbf{R}^n$. Prove that $f(x) = 0$ almost everywhere.

   Hint. Show that there exists a constant $c = c(f)$ such that $Mf(x) \geq c(f)|x|^{-n}$ whenever x is sufficiently large.

2. **Problem 2.** Let $f(x) = 1$ if $x \in [0,1]$, and zero otherwise. Define $h_c(x) := \sup_{n\in\mathbf{N}} n^c f(nx)$. Prove that

   a) h_c is integrable on $\mathbf{R}$ if $c \in (0,1)$.

   b) h_1 is in weak-type L but not integrable on $\mathbf{R}$.

   c) h_c is not in weak-type L if $c > 1$.

   Hint. For a.e. $x \in \mathbf{R}$, $h_c(x) = \sum\limits_{n=1}^{\infty} 1_{(1/(n+1),1/n]}(x)n^c$.

3. **Problem 3.** For any set $E \subset \mathbf{R}^2$, the boundary ∂E of E is, by definition, the closure of E minus the interior of E.

   a) Show that E is Lebesgue measurable whenever $m(\partial E) = 0$.

   b) Suppose that E is the union of a (*possibly uncountable*) collection of **closed** discs in $\mathbf{R}^2$ whose radii are at least 1 and at most 2. Use a) to show that E is Lebesgue measurable.

   Hint. Observe that the density of the boundary points is always < 1. On the other hand, if $m(\partial E) > 0$, then the Lebesgue theorem gives 1 for almost all $x \in \partial E$.

   c) Show that the conclusion of b) is true even when the radii are unrestricted.

   d) Show that some unions of closed discs of radius 1 are not Borel sets.

   Hint. What if all the discs touch the straight line?

4. **Problem 4.** Let f be periodic with the period 1 and Lebesgue integrable with respect to the Lebesgue measure on $[0,1]$. Define the **Riemann sum**

$$f_n(x) := \frac{1}{n}\sum_{k=1}^{n} f(x + \frac{k}{n}), \qquad n \in \mathbf{N}.$$

ℵ) Prove that a sequence $(f_{2^n}(x))_{n=1}^{\infty}$ converges almost everywhere to $A = \int_0^1 f(x)dx$ by completing the following steps.

a) Assume that $f \geq 0$. Show that $s_N f(x) := \sup_{0 \leq n \leq N} f_{2^n}(x)$ is in weak type. More precisely, prove that

$$\forall \lambda > 0 \qquad \lambda m(E_\lambda) \leq \int_{E_\lambda} f(x)dx, \qquad E_\lambda := \{x \in [0,1] : s_N f(x) > \lambda\}.$$

Hint. Define $E_n := \{x \in [0,1] : f_{2^n}(x) > \lambda\}$, $0 \leq n \leq N$, $N \in \mathbf{N}$, and observe that

$$E_\lambda = \cup_{n=1}^{N} E_n = \cup_{n=1}^{N} A_n, \qquad A_N := E_N, \qquad A_n := E_n \setminus (\cup_{i=n+1}^{N} E_i), \ n = 0, ..., N-1,$$

where A_n are disjoint. Prove that all A_n are periodic with period 2^{-n}, and conclude that

$$\int_{A_n} f(x)dx = \int_{A_n} f_{2^n}(x)dx \geq \lambda m(A_n).$$

This implies ℵ).

b) Letting $N \to \infty$ prove that

$$\lambda m(\{x \in [0,1] : \sup_{n \in \{0\} \cup \mathbf{N}} f_{2^n}(x) > \lambda\}) \leq \int_0^1 f(x)dx.$$

c) Define $\Phi(x) := \overline{\lim}_{n\to\infty} f_{2^n}(x)$. Prove that Φ has arbitrary small periods, hence $\Phi(x) = const$ almost everywhere.

Hint. Use Problem 1, d) of Assignment XI.

d) Prove that $\Phi \leq A$ by taking any $\lambda < \Phi$. Conclude that $\Phi = A$.

Hint. Replace $f(x)$ by $-f(x)$.

ב) Show that in fact any sequence $(f_{m_n}(x))_{n=1}^{\infty}$ converges to A almost everywhere, provided m_{n-1} divides m_n.

ג) Find an integrable periodic function f such that a sequence of Riemann sums $(f_n(x))_{n=1}^{\infty}$ diverges almost everywhere.

Hint. Assume (do not prove) that for every **irrational** x there are infinitely many integers n such that $|x - m/n| < 1/n^2$ for some integer m.

Fix $0 < \alpha < 1/2$. Take $f(x) = |x|^{-1+\alpha}$ for $|x| \leq 1/2$, and define $f(x)$ for all other x by periodicity. Then $f(x - m/n) > f(1/n^2) = n^{2-2\alpha}$, so $f_n(x) > n^{1-2\alpha}$.