# Real Analysis, Math 821.

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# Assignment XIV.

1. **Problem 1.**

   a) Let $f(x) = x^\alpha$, $x \in (0,1]$, $\alpha \in \mathbf{R}$. Describe $\alpha$ for which $f \in L^p(m, [0,1])$ for (fixed) $p \geq 1$.

   b) Let $\mu(X) = +\infty$, and let $p \geq 1$. Construct a function $f \in L^p(\mu, X)$ such that $f \notin L^r(\mu, X)$ for $r \geq 1$, $r \neq p$.

   c) Let $\mu(X) = +\infty$, and let $1 \leq s < p < \infty$. Prove that $f \in L^s(\mu, X)$, $f \in L^p(\mu, X)$ implies $f \in L^r(\mu, X)$ for $s < r < p$.

   d) Let $\mu(X) = +\infty$. Construct a function $f \notin L^\infty(\mu, X)$, but $f \in L^p(\mu, X)$ $\forall p \geq 1$.

2. **Problem 2.**

   a) Prove that convex function on $(a,b)$ is continuous on $(a,b)$.

   **Hint**. See Rudin, page 61.

   b) Prove that the supremum of any collection of convex functions on $(a,b)$ is convex (if it is finite) and that pointwise limits of sequences of convex functions are convex. What can you say about upper and lower limits of sequences of convex functions?

   c) If $\phi$ is convex on $(a,b)$, and if $\psi$ is convex and nondecreasing on the range of $\phi$, prove that $\psi(\phi)$ is convex on $(a,b)$. For $\phi > 0$ show that the convexity of $\log \phi$ implies the convexity of $\phi$, but not vice versa.

   d) Let $f$ be convex on $(a,b)$. Prove that $f$ satisfies the **Jensen inequality**,

   $$f(\sum_{j=1}^{n} \lambda_j x_j) \leq \sum_{j=1}^{n} \lambda_j f(x_j), \qquad \forall x_j \in (a,b),\ \lambda_j \in [0,1],\ \sum_{j=1}^{n} \lambda_j = 1.$$

   **Hint**. Prove at first that $\sum_{j=1}^{n} \lambda_j x_j \in (a,b)$. Then use induction.

   e)* Prove that if $f$ is **continuous** and satisfies $f((x+y)/2) \leq (f(x)+f(y))/2$, $\forall x, y \in (a,b)$, then $f$ is convex.

   **Hint**. Use d) for **rational** $\lambda_j$.

3. **Problem 3.** Let $K$ be a convex body in $\mathbf{R}^n$, and let $x \cdot y := \sum_{k=1}^{n} x_k y_k$ be a usual inner product in $\mathbf{R}^n$. Define the **polar body** $K^*$ of $K$ as $K^* := \{x \in \mathbf{R}^n : x \cdot y \leq 1\ \forall y \in K\}$, and the $l_n^p$**-balls**, $1 \leq p < \infty$, as $B_n^p := \{x \in \mathbf{R}^n : \sum_{k=1}^{n} |x_k|^p \leq 1\}$, $B_n^\infty := \{x \in \mathbf{R}^n : \sup_{k=1,\dots,n} |x_k| \leq 1\}$.

a) Prove that $\left(B_n^2\right)^* = B_n^2$.

b) Let $K := \{x \in \mathbf{R}^n : \sum\limits_{k=1}^{n} |x_k|^2/a_k^2 \leq 1,\ a_k > 0\}$ be an ellipsoid. Find $K^*$.

c) Prove that $\left(B_n^p\right)^* = B_n^q$, where $1/p + 1/q = 1$.

**Hint**. You might use the Lagrange multipliers.

4. **Problem 4.** Suppose $\mu$ is a positive measure on $X$, $\mu(X) < \infty$, $f \in L^\infty(\mu, X)$, $\|f\|_\infty > 0$, and $\alpha_n := \int\limits_X |f(x)|^n d\mu$, $n \in \mathbf{N}$. Prove that $\lim\limits_{n\to\infty} \alpha_{n+1}/\alpha_n = \|f\|_\infty$.

   **Hint**. Prove at first that $\|f\|_n \to \|f\|_\infty$ as $n \to \infty$. Then use (and prove) the fact that for **any** sequence of positive numbers $(\alpha_n)_{n=1}^\infty$, $\lim\limits_{n\to\infty} \alpha_{n+1}/\alpha_n = \lim\limits_{n\to\infty} \alpha_n^{1/n}$, provided the first limit exists.