Real Analysis, Math 821.

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Assignment II.

1. **Problem 1.** Let S be a collection of sets. Prove that there exists one and only one **minimal ring** $R(S)$, such that $S \subset R(S)$. (This, in particular, means that for any ring R containing S, $R(S) \subset R$).

   Hint. Define $X := \cup_{A \in S} A$, and consider $P(X)$ (the set of all subsets of X). Let Σ be a collection of all rings containing S and which are contained in $P(X)$. The intersection of all of these rings (elements of Σ) will be your **minimal ring**.

2. **Problem 2*.** Let S be a subring. Prove that the minimal ring $R(S)$ is a collection of sets of the type $A = \cup_{k=1}^{n} A_k$, $A_k \in S$.

   Hint. Prove by induction (and then use) the following statement.

   Lemma. Let A_i, A, $i = 1, ..., n$ belong to a subring S, $A_i \subset A$, and let $A_i \cap A_j = \varnothing$ for $i \neq j$. Then there are sets $A_{n+1}, ...A_s$ belonging to S, such that $A = \cup_{k=1}^{s} A_k$, $s \geq n$.

3. **Problem 3.** Let $\Big(E_n\Big)_{n=1}^{\infty}$ be a sequence of sets. The **set** $\overline{\lim\limits_{n\to\infty}} E_n := \cap_{n=1}^{\infty}\Big(\cup_{k=n}^{\infty} E_k\Big)$ is called the upper limit of the sequence. The **set** $\underline{\lim\limits_{n\to\infty}} E_n := \cup_{n=1}^{\infty}\Big(\cap_{k=n}^{\infty} E_k\Big)$ is called the lower limit of the sequence. Prove that $\underline{\lim\limits_{n\to\infty}} E_n \subseteq \overline{\lim\limits_{n\to\infty}} E_n$. If they are equal, then their value (the set) is called a **limit of sequence** of sets.

4. **Problem 4.** Give an example of sets for which $\underline{\lim\limits_{n\to\infty}} E_n \neq \overline{\lim\limits_{n\to\infty}} E_n$.

5. **Problem 5.** Let X be a set, and let $\Big(E_n\Big)_{n=1}^{\infty}$ be a sequence of sets satisfying $E_n \subset X$ for $n = 1, ..., \infty$. Prove

$$X \setminus \overline{\lim_{n\to\infty}} E_n = \underline{\lim_{n\to\infty}} \Big(X \setminus E_n\Big).$$

6. **Problem 6.** Let $\Big(E_n\Big)_{n=1}^{\infty}$ be a sequence of sets and let $\Big(1_{E_n}\Big)_{n=1}^{\infty}(x)$ be a sequence of their characteristic functions. Prove that

$$1_{\{\overline{\lim_{n\to\infty}} E_n\}}(x) = \overline{\lim_{n\to\infty}} 1_{E_n}(x), \qquad 1_{\{\underline{\lim_{n\to\infty}} E_n\}}(x) = \underline{\lim_{n\to\infty}} 1_{E_n}(x).$$