Real Analysis, Math 821.

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Assignment II.

1. **Problem 1.** Let $S$ be a collection of sets. Prove that there exists one and only one minimal ring $R(S)$, such that $S \subseteq R(S)$. (This, in particular, means that for any ring $R$ containing $S$, $R(S) \subseteq R$).

   **Hint.** Define $X := \bigcup_{A \in S} A$, and consider $P(X)$ (the set of all subsets of $X$). Let $\Sigma$ be a collection of all rings containing $S$ and which are contained in $P(X)$. The intersection of all of these rings (elements of $\Sigma$) will be your minimal ring.

2. **Problem 2.** Let $S$ be a subring. Prove that the minimal ring $R(S)$ is a collection of sets of the type $A = \bigcup_{k=1}^{s} A_k$, $A_k \in S$.

   **Hint.** Prove by induction (and then use) the following statement.

   **Lemma.** Let $A_i, A, i = 1, \ldots, n$ belong to a subring $S$, $A_i \subseteq A$, and let $A_i \cap A_j = \emptyset$ for $i \neq j$. Then there are sets $A_{n+1}, \ldots, A_s$ belonging to $S$, such that $A = \bigcup_{k=1}^{s} A_k$, $s \geq n$.

3. **Problem 3.** Let $\left( E_n \right)_{n=1}^{\infty}$ be a sequence of sets. The set $\liminf_{n \to \infty} E_n := \bigcap_{n=1}^{\infty} \left( \bigcup_{k=n}^{\infty} E_k \right)$ is called the upper limit of the sequence. The set $\limsup_{n \to \infty} E_n := \bigcup_{n=1}^{\infty} \left( \bigcap_{k=n}^{\infty} E_k \right)$ is called the lower limit of the sequence. Prove that $\limsup_{n \to \infty} E_n \subseteq \liminf_{n \to \infty} E_n$. If they are equal, then their value (the set) is called a limit of sequence of sets.

4. **Problem 4.** Give an example of sets for which $\lim_{n \to \infty} E_n \neq \liminf_{n \to \infty} E_n$.

5. **Problem 5.** Let $X$ be a set, and let $\left( E_n \right)_{n=1}^{\infty}$ be a sequence of sets satisfying $E_n \subseteq X$ for $n = 1, \ldots, \infty$. Prove

\[ X \setminus \lim_{n \to \infty} E_n = \lim_{n \to \infty} \left( X \setminus E_n \right). \]

6. **Problem 6.** Let $\left( E_n \right)_{n=1}^{\infty}$ be a sequence of sets and let $\left( 1_{E_n} \right)_{n=1}^{\infty}(x)$ be a sequence of their characteristic functions. Prove that

\[ 1_{\lim_{n \to \infty} E_n}(x) = \lim_{n \to \infty} 1_{E_n}(x), \quad 1_{\liminf_{n \to \infty} E_n}(x) = \lim_{n \to \infty} 1_{E_n}(x). \]