Real Analysis, Math 821.

Instructor: Dmitry Ryabogin

Assignment III.

1. **Problem 1.**

   Definition 1. Let $E := [0,1] \times [0,1]$, and let $A \subset E$. The **inner measure** $\mu_*(A)$ (of A) is a number defined as
   $$\mu_*(A) := 1 - \mu^*(E \setminus A).$$
   Prove that $\mu_*(A) \le \mu^*(A)$.

   Hint. $\quad \mu^*(E \setminus A) + \mu^*(A) \ge \mu^*(E)$.

2. **Problem 2.** Prove that $A \subseteq E$ is Lebesgue measurable if and only if $\mu_*(A) = \mu^*(A)$.

   Hint. Let $A \subseteq E$ be Lebesgue measurable. Then $\forall \epsilon > 0$ there exists an elementary set B such that $\mu^*(A \triangle B) < \epsilon$. Prove that
   $$\mu^*(B) - \epsilon \le \mu_*(A) \le \mu^*(A) \le \mu^*(B) + \epsilon.$$
   Conversely, assume that $\mu_*(A) = \mu^*(A)$. Then prove the following chain of statements leading to the result.

   a) $\forall \epsilon > 0$ there exist elementary sets P_n, Q_n, $n = 1, 2, ...$, such that
   $$A \subset \cup_{n=1}^{\infty} P_n, \qquad \mu^*(A) \ge \sum_{n=1}^{\infty} m'(P_n) - \epsilon,$$
   and
   $$(E \setminus A) \subset \cup_{n=1}^{\infty} Q_n, \qquad \mu^*(E \setminus A) \ge \sum_{n=1}^{\infty} m'(Q_n) - \epsilon.$$
   Conclude that
   $$\sum_{n=1}^{\infty} (m'(P_n) + m'(Q_n)) \le 1 + 2\epsilon,$$
   and that there exists N such that
   $$\sum_{n=N+1}^{\infty} (m'(P_n) + m'(Q_n)) < \epsilon.$$

   b) Denote
   $$\cup_{n=1}^{\infty} P_n = P, \qquad \cup_{n=1}^{\infty} Q_n = Q, \qquad \cup_{n=1}^{N} P_n = P_N \qquad \cup_{n=1}^{N} Q_n = Q_N.$$
   Observe that
   $$\mu^*(P_N \triangle A) \le \mu^*(P_N \setminus A) + \mu^*(A \setminus P_N).$$

c) Prove that

$$\mu^*(A \setminus P_N) \leq \sum_{n=N+1}^{\infty} m'(P_n).$$

d) Observe that

$$P_N \setminus A \subset (P_N \cap Q_N) \cup (P_N \cap (Q \setminus Q_n)) \subset (P_N \cap Q_N) \cup (Q \setminus Q_n),$$

and conclude

$$\mu^*(P_N \setminus A) \leq \mu^*(P_N \cap Q_N) + \sum_{n=N+1}^{\infty} m'(Q_n).$$

e) Observe that $E \subseteq (P \cup Q)$, and show that

$$1 = \mu^*(E) \leq \mu^*(P_N \cup Q_N) + \mu^*(P \setminus P_N) + \mu^*(Q \setminus Q_N).$$

Use the fact that for elementary sets C, D,

$$m'(C \cup D) = m'(C) + m'(D) - m'(C \cap D),$$

and a) to conclude that

$$1 \leq \sum_{n=1}^{\infty}(m'(P_n) + m'(Q_n)) - \mu^*(P_N \cap Q_N) \leq 1 + 3\epsilon - \mu^*(P_N \cap Q_N),$$

and $\mu^*(P_N \cap Q_N) \leq 3\epsilon$.

f) Use d) to show that

$$\mu^*(P_N \setminus A) \leq 2\epsilon + \sum_{n=N+1}^{\infty} m'(Q_n).$$

g) "Glue" pieces b), c), f) to obtain the desired result.

3. **Problem 3.** Prove that the capacity of all Lebesgue measurable sets is greater that continuum.

   Hint. Consider $P(C)$ (the set of all subsets of the Cantor set).

4. **Problem 4*.** Let $A \subset [0, 1]$ be such that in the decimal decomposition of every $x \in A$ you meet 2 before you meet 3. Find the Lebesgue measure of A.

   Hint. This is "Cantor-like" exercise. Think first about numbers you would not like to have (go for the complement). At first take out of $[0, 1]$ the set $[0.3, 0.4]$. Then take out eight (why not nine?) sets of the type $[0.n_1 3, 0.n_1 4]$, where $n_1 = 0, 1, 4, 5, 6, 7, 8, 9$, and so on...

5. **Problem 5.** Let $E = [0,1] \times [0,1]$ be a unit square in the plane, and let

$$A := \{(x,y) \in E : |\sin x| < \frac{1}{2},\ \cos(x+y) \in \mathbf{R} \setminus \mathbf{Q}\}.$$

Find the Lebesgue measure of A.

Hint. What is the complement of A?

6. **Problem 6*.** A set $A \subseteq E$ is called **Caratheodory measurable** if $\forall Z \subseteq E$ we have

$$\mu^*(Z) = \mu^*(Z \cap A) + \mu^*(Z \setminus A).$$

Prove that A is Caratheodory measurable if and only if it is Lebesgue measurable.

Hint. Let A be Caratheodory measurable. Then by taking $Z = E$, we have

$$\mu^*(E) = \mu^*(A) + \mu^*(E \setminus A) = \mu^*(A) + \mu_*(A).$$

This gives (see Problem 2) the Lebesgue measurability of A.

Conversely, let A be Lebesgue measurable. Then $\forall Z \subseteq E$, we have

$$\mu^*(Z) \le \mu^*(Z \cap A) + \mu^*(Z \setminus A),$$

and it remains to prove

$$\mu^*(Z) \ge \mu^*(Z \cap A) + \mu^*(Z \setminus A).$$

To this end, prove the following (**important**)

Lemma. $\forall Z \subseteq E$ there exists a **Lebesgue measurable** set Z_1 such that $Z \subseteq Z_1 \subseteq E$, and $\mu^*(Z) = \mu^*(Z_1)$.

Then conclude that

$$\mu^*(Z) = \mu^*(Z_1) = \mu^*(Z_1 \cap A) + \mu^*(Z \setminus A) \ge \mu^*(Z \cap A) + \mu^*(Z \setminus A).$$