# Real Analysis, Math 821.

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# Assignment IV.

1. **Problem 1.**

   **Definition 1.** A set $M \subset \mathbf{R}$ is called closed if it coincides with its closure. In other words, $M \subset \mathbf{R}$ is closed, if it contains all of its limit points.

   **Definition 2.** A set $A \subset \mathbf{R}$ is called open if there is a closed set $M \subset \mathbf{R}$ such that $A = \mathbf{R} \setminus M$. In other words, $A$ is open if all of its points are **inner** points, (a point $x$ is called the inner point of a set $A$, if there exists $\epsilon > 0$, such that the $\epsilon$-neighbourhood $U_\epsilon(x)$ of $x$, is contained in $A$, $U_\epsilon(x) \subset A$).

   Prove that any open set on the real line is a finite or infinite union of disjoint open intervals. (The sets $(-\infty, \infty)$, $(\alpha, \infty)$, $(-\infty, \beta)$ are intervals for us).

   **Hint**. Let $G$ be open on $\mathbf{R}$. Introduce in $G$ the equivalence relationship, by saying that $x \sim y$ if there exists an interval $(\alpha, \beta)$, such that $x, y \in (\alpha, \beta) \subset G$.

   a) Prove at first that this is indeed the equivalence relationship, i. e., that $x \sim x$, $x \sim y$ implies $y \sim x$, and $x \sim y$, $y \sim z$ imply $x \sim z$. Conclude that $G$ can be written as a disjoint union $G = \cup_{\tau \in \beth} I_\tau$ of "classes" of points $I_\tau$, $I_\tau = \{x \in G : x \sim \tau\}$, $\beth$ is the set of different indices, i.e. $\tau \in \beth$ if $I_\tau \cap I_x = \varnothing$, $\forall x \in G$.

   b) Prove that each $I_\tau$ is an open interval.

   c) Show that there are at most countably many $I_\tau$ (the set $\beth$ is countable).

2. **Problem 2.**

   **Definition 3.** A set $K \subset \mathbf{R}$ is called **nowhere dense** in $\mathbf{R}$, if for any open interval $(\alpha, \beta)$ there exists an open interval $(\gamma, \delta) \subset (\alpha, \beta)$, such that $K \cap (\gamma, \delta) = \varnothing$.

   a) Prove that a **closed** set $K \subset \mathbf{R}$ is nowhere dense, if and only if for any interval $(\alpha, \beta)$ there exists a point $x \in (\alpha, \beta) \setminus K$.

   b) Prove that the Cantor set is nowhere dense in $\mathbf{R}$.

   c) Let $B$ be an **open** dense (the closure of $B$ is $\mathbf{R}$) subset of the real line. Is it true that $\mathbf{R} \setminus B$ is nowhere dense?

   d) Let $D$ be a dense subset of the real line. Is it true that $\mathbf{R} \setminus D$ is nowhere dense?

   e) Prove that the closure of a nowhere dense set is nowhere dense.

3. **Problem 3.** a) Let $(r_k)_{k=1}^\infty$ be a sequence of all rational numbers on the real line, and let $v_k = (r_k - k^{-2}, r_k + k^{-2})$. Prove that $E := \mathbf{R} \setminus (\cup_{k=1}^\infty v_k)$ is closed and nowhere dense.

   **Definition 3.** A set $K \subset \mathbf{R}$ is called **of first category** if it is a countable union of nowhere dense sets.

b) Prove that no interval in $\mathbf{R}$ is of the first category.

**Hint**. Our statement is an obvious consequence of the following

**Lemma**. The complement of any set $A$ of the first category on the line is dense.

To prove the Lemma you could argue as follows. Let $A = \cup_{i=1}^{\infty} A_i$, where $A_i$ are nowhere dense. Let $I$ be any interval on the real line, and let $I_1$ be a closed subinterval of $I \setminus A_1$, $I_2$ be a closed subinterval of $I_1 \setminus A_2$, and so on. Show that

$$\varnothing \neq \cap_{n=1}^{\infty} \subset (I \setminus A).$$

c)* Prove that $\mathbf{R}$ can be written as a disjoint union of $A \cup (\mathbf{R} \setminus A)$, where $|A| = 0$, and $\mathbf{R} \setminus A$ is of first category.

**Hint**. Let $(r_k)_{k=1}^{\infty}$ be a sequence of all rational numbers on the real line, and let $I_{kj} = (r_k - 2^{-k-j}, r_k + 2^{-k-j})$. Define

$$G_j := \cup_{k=1}^{\infty} I_{kj}, \qquad A := \cap_{j=1}^{\infty} G_j.$$

Show that $|A| < \epsilon$, $\forall \epsilon > 0$. On the other hand, $G_j$ is open (why?) and dense (why?), hence $\mathbf{R} \setminus G_j$ is nowhere dense (Problem 2, c)).

4. **Problem 4.**

**Definition 4.** A set $M \subset \mathbf{R}$ is called **perfect**, if it is closed, bounded, and has no isolated points. Any closed interval is perfect. Cantor set is perfect (why?). A set containing two different points is not perfect.

a) Prove that $M$ is perfect if and only if disjoint intervals from $\mathbf{R} \setminus M$ do not have "common ends" (intervals $(a, b), (b, c)$ have common ends, intervals $(a, b), (c, d)$, $b < c$, do not have).

b) Construct a nonempty perfect nowhere dense susbset of $\mathbf{R} \setminus \mathbf{Q}$.

**Hint**. Let $(r_k)_{k=1}^{\infty}$ be a sequence of all rational numbers on the real line. Take $r_1$, and let $\epsilon_1$ be any irrational number. Define $I_1 := U_{\epsilon_1}(r_1)$. Now take the first point from $(r_k)_{k=2}^{\infty} \setminus I_1$. Let it be $r_{k_2}$. Observe that $r_{k_2} \neq r_1 \pm \epsilon_1$ (why?). Now take $\epsilon_2 \in \mathbf{R} \setminus \mathbf{Q}$ such that $U_{\epsilon_2}(r_{k_2}) \cap I_1 = \varnothing$, and such that $U_{\epsilon_2}(r_{k_2})$, $I_1$ have no common ends. Define $I_2 := U_{\epsilon_2}(r_{k_2})$. On the next step find among $r_{k_2+1}, r_{k_2+2}, \ldots$, the first point out of $I_1 \cup I_2$. Let it be $r_{k_3}$. Observe that $r_{k_3} \neq r_1 \pm \epsilon_1$, $r_{k_3} \neq r_{k_2} \pm \epsilon_2$. Now find $\epsilon_3 \in \mathbf{R} \setminus \mathbf{Q}$ such that $U_{\epsilon_3}(r_{k_3}) \cap I_1, I_2 = \varnothing$, and such that $U_{\epsilon_3}(r_{k_3})$, $I_1$, $I_2$ have no common ends. Define $I_3 := U_{\epsilon_3}(r_{k_3})$.

Continue this way to obtain a sequence $(I_k)_{k=1}^{\infty}$. Then $E := \mathbf{R} \setminus (\cup_{k=1}^{\infty} I_k)$ is the set you are looking for. Prove that $E$ is closed. Prove that $E \neq \varnothing$. Prove that $E$ does not contain isolated points. Prove that $E$ does not contain any rational point. Finally, prove that $E$ is nowhere dense (use the closeness of $E$ and Problem 2, a)).

5. **Problem 5.** a) Is it possible to construct a countable family $(E_i)_{i=1}^{\infty}$ of perfect nowhere dense sets $E_i \subset [a, b]$ such that $| \cup_{i=1}^{\infty} E_i| = b - a$?

**Hint**. Take $E_i \subset [a, b]$, such that $|E_i| = b - a - 1/i$, and $E_i$ is perfect and nowhere dense (how to construct these sets?). To show that $\cup_{i=1}^{\infty} E_i$ satisfies the requirements of the problem define $A_i := \cup_{k=1}^{i} E_i$, and observe that $\cup_{i=1}^{\infty} E_i = \cup_{i=1}^{\infty} A_i$.

b)* Is it possible to construct a countable family $(E_i)_{i=1}^{\infty}$ of **pairwise disjoint** perfect nowhere dense sets $E_i \subset [a, b]$ such that $| \cup_{i=1}^{\infty} E_i| \equiv \sum_{i=1}^{\infty} |E_i| = b - a$?

**Hint**. Consider the case $[a, b] = [0, 1]$. Construct a perfect nowhere dense set $E_1 \subset [0, 1]$, such that $|E_1| = 1/2$. Take any $(a_{1,i}, b_{1,i}) \subset ([0, 1] \setminus E_1)$ and put there a perfect nowhere dense set $E_{1,i} \subset (a_{1,i}, b_{1,i})$, such that $|E_{1,i}| = (b_{1,i} - a_{1,i})/2$. Then $| \cup_{i=1}^{\infty} E_{1,i}| = 1/2^2$. Observe that $E_2 := E_1 \cup (\cup_{i=1}^{\infty} E_{1,i})$ is perfect and nowhere dense, moreover, $|E_2| = 3/4$.

Take any $(a_{2,i}, b_{2,i}) \subset ([0, 1] \setminus E_2)$ and put there a perfect nowhere dense set $E_{2,i} \subset (a_{2,i}, b_{2,i})$, such that $|E_{2,i}| = (b_{2,i} - a_{2,i})/2$. Then $| \cup_{i=1}^{\infty} E_{2,i}| = 1/2^3$. Observe that $E_3 := E_2 \cup (\cup_{i=1}^{\infty} E_{2,i})$ is perfect and nowhere dense, moreover, $|E_3| = 1 - 1/2^3$.

Continue this way, and get a sequence of perfect nowhere dense sets

$$E_1; \qquad E_{1,1}, E_{1,2}, E_{1,3}, ..., E_{1,i}, ...; \qquad E_{2,1}, E_{2,2}, E_{2,3}, ..., E_{2,i}, ...;$$

$$E_{k,1}, E_{k,2}, E_{k,3}, ..., E_{k,i}, ...; ...$$

Observe that all of them are disjoint, and make the desired conclusion.