1. **Problem 1.** Let $E = [0, 1] \times [0, 1] \subset \mathbb{R}^2$, and let $S$ be a subring of rectangles of type $T_{ab} := \{a \leq x < b, \ 0 \leq y \leq 1\}$. Define $m(T_{ab}) := b - a$.

a) Describe the Lebesgue continuation of this measure. What sets are going to be measurable?

b) Prove that $\tilde{T} := \{0 \leq x \leq 1, \ y = 1/2\}$ is not measurable, and find its outer measure.

**Hint.** The set is Lebesgue measurable if and only if its outer measure is equal to its inner measure.

2. **Problem 2.**

**Definition 1.** Let $U$ be a collection of all open subsets of the real line. Then $R(U)$ is called Borel sets (the minimal ring containing $U$).

Prove that any Lebesgue measurable set on the real line is a union of a Borel set and a set of measure zero.

**Hint.** Let $A \subset \mathbb{R}$ be measurable. According to Assignment III, Problem 2, $\forall \epsilon > 0$, there exists a closed set $B_\epsilon \subset A$ such that $\mu^*(A \setminus B_\epsilon) < \epsilon$. The set you are looking for is $\bigcup_{n=1}^{\infty} B_{1/n}$.

3. **Problem 3.**

**Definition 2.** We say that a measure $\mu$ (defined on a corresponding subring $S$) is invariant under the transformation $T : S \to S$ if

\[ \forall A \in S, \quad \mu(T^{-1}(A)) = \mu(A). \]

a) It is known (take it as granted) that a real number $x \in [0, 1]$ can be written as a continuous fraction

\[ x = \frac{1}{n_1 + \frac{1}{n_2 + \ldots}}, \quad n_k \in \mathbb{N}, \]

where a rational number can be written as a finite fraction, and an irrational number as an infinite one. Define the transformation $T$ on $[0, 1]$ as $T := \{1/x\}$, where $\{\cdot\}$ stands for the fractional part of a number. Prove that (in terms of sequences $(n_k)_{k=1}^{\infty}$), $T$ has the form $T((n_k)_{k=1}^{\infty}) = (n_{k+1})_{k=1}^{\infty}$.

b) Let $\mu$ be a measure on $[0, 1]$, defined as

\[ \mu((\alpha, \beta)) := \log_2 \frac{1 + \beta}{1 + \alpha}. \]

Prove that $\mu$ is invariant under $T$ defined in a).
4. Problem 4. Let \( m \) be a measure on a subring \( S \), and let \( \mu \) be its extension to \( R(S) \). Prove that the following statements are equivalent for \( \mu \), and might be not equivalent for \( m \).

\( \& \) \( \sigma \)-additivity.
\[
\mu \left( \bigcup_{k=1}^{\infty} A_k \right) = \sum_{k=1}^{\infty} \mu(A_k);
\]

\( \exists \) upper semicontinuity.
\[
A_1 \supset A_2 \supset A_3 \supset \ldots, \quad A = \bigcap_{k=1}^{\infty} A_k, \quad \Rightarrow \quad \mu(A) = \lim_{k \to \infty} \mu(A_k);
\]

\( \exists \) lower semicontinuity.
\[
A_1 \subset A_2 \subset A_3 \subset \ldots, \quad A = \bigcup_{k=1}^{\infty} A_k, \quad \Rightarrow \quad \mu(A) = \lim_{k \to \infty} \mu(A_k);
\]

\( \exists \) continuity.
\[
\mu(\lim_{k \to \infty} A_k) = \lim_{k \to \infty} \mu(A_k).
\]

**Hint.** Prove that \( \& \) \( \iff \) \( \exists \), \( \& \iff \exists \), \( \exists \implies \& \). Prove that \( \exists \), \( \exists \) imply
\[
\mu(\lim_{n \to \infty} A_n) = \mu(\bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} A_n) = \lim_{k \to \infty} \mu(\bigcup_{n=k}^{\infty} A_n) \geq \lim_{k \to \infty} \mu(A_k),
\]

\[
\mu(\lim_{n \to \infty} A_n) = \mu(\bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} A_n) = \lim_{k \to \infty} \mu(\bigcap_{n=k}^{\infty} A_n) \leq \lim_{n \to \infty} \mu(A_k).
\]

This will give you \( \& \iff \exists \).

Moreover, consider the following example of a measure, which is not \( \sigma \)-additive. Take a subring \( S \) of subsets of \([0, 1) \cap \mathbb{Q}\), and define
\[
S := \{ s_{a,b} := [a, b) \cap [0, 1) \cap \mathbb{Q} \}, \quad m(s_{a,b}) = b - a.
\]

Prove that for \( m \), \( \exists \) and \( \exists \) are true, but \( \& \) and \( \exists \) are not. On the other hand for the extension \( \mu \) of \( m \), all \( \& \), \( \exists \), \( \exists \), \( \exists \) are not true (they are equivalent).

5. Problem 5.

**Definition 3.** A pair \((X, d)\) is called a **metric space**, if \( X \) is a set, and \( d \) is a distance. More precisely, \( d \) is a nonnegative real function \( d(x, y) \) defined for any \( x, y \in X \), and satisfying

1) \( d(x, y) = 0 \iff x = y \),

2) \( d(x, y) = d(y, x) \) (the axiom of symmetry)

3) \( d(x, z) \leq d(x, y) + d(y, z) \) (the axiom of triangle).

Let \( \mu \) be a \( \sigma \)-additive measure on a subring \( S \subset P(X) \), and let \( \mu^* \) be the corresponding outer measure on \( P(X) \).

a) We say that \( A \sim B \) if \( \mu^*(A \Delta B) = 0 \). Prove that this is an equivalence relation.

b) Let \( \tilde{X} \) be a set of all classes \( \tilde{A} \) of equivalence. Prove that \((\tilde{X}, d)\) is a metric space, where \( d(\tilde{A}, \tilde{B}) := \mu^*(A \Delta B) \). Here \( \tilde{A}, \tilde{B} \) are classes of equivalence containing \( A, B \).
6. **Problem 6.** Is it possible to construct a set $G \subset [0, 1]$ such that

\begin{enumerate}
  \item $G$ is dense on $[0, 1]$,
  \item $G$ has measure (length) zero,
  \item $G$ is not countable?
\end{enumerate}

**Hint.** Consider $G := [0, 1] \setminus E$, where $E$ is a union of sets $E_i$ constructed in the previous Assignment, Problem 5, b)*.