Real Analysis, Math 821.

Instructor: Dmitry Ryabogin

Assignment V.

1. **Problem 1.** Let $E = [0,1] \times [0,1] \subset \mathbf{R^2}$, and let S be a subring of rectangles of type $T_{ab} := \{a \leq x < b,\ 0 \leq y \leq 1\}$. Define $m(T_{ab}) := b - a$.

   a) Describe the Lebesgue continuation of this measure. What sets are going to be measurable?

   b) Prove that $\tilde{T} := \{0 \leq x \leq 1,\ y = 1/2\}$ is not measurable, and find its outer measure.

   Hint. The set is Lebesgue measurable if and only if its outer measure is equal to its inner measure.

2. **Problem 2.**

   Definition 1. Let $\mathbf{U}$ be a collection of all open subsets of the real line. Then $R(\mathbf{U})$ is called **Borel sets** (the minimal ring containing $\mathbf{U}$).

   Prove that any Lebesgue measurable set on the real line is a union of a Borel set and a set of measure zero.

   Hint. Let $A \subset \mathbf{R}$ be measurable. According to Assignment III, Problem 2, $\forall \epsilon > 0$, there exists a closed set $B_\epsilon \subset A$ such that $\mu^*(A \setminus B_\epsilon) < \epsilon$. The set you are looking for is $\cup_{n=1}^{\infty} B_{1/n}$.

3. **Problem 3.**

   Definition 2. We say that a measure μ (defined on a corresponding subring S) is invariant under the transformation $\mathbf{T} : S \to S$ if

   $$\forall A \in S, \qquad \mu(\mathbf{T^{-1}}(A)) \equiv \mu(A).$$

   a) It is known (take it as granted) that a real number $x \in [0,1]$ can be written as a **continuous fraction**

   $$x = \frac{1}{n_1 + \frac{1}{n_2 + \ldots}}, \qquad n_k \in \mathbf{N},$$

   where a rational number can be written as a finite fraction, and an irrational number as an infinite one. Define the transformation $\mathbf{T}$ on $[0,1]$ as $\mathbf{T} := \{1/x\}$, where $\{\cdot\}$ stands for the fractional part of a number. Prove that (in terms of sequences $(n_k)_{k=1}^{\infty}$), $\mathbf{T}$ has the form $\mathbf{T}((n_k)_{k=1}^{\infty}) = (n_{k+1})_{k=1}^{\infty}$.

   b) Let μ be a measure on $[0,1]$, defined as

   $$\mu([\alpha, \beta)) := \log_2 \frac{1+\beta}{1+\alpha}.$$

   Prove that μ is invariant under $\mathbf{T}$ defined in a).

4. **Problem 4.** Let m be a measure on a subring S, and let μ be its extension to $R(S)$. Prove that the following statements are equivalent for μ, and might be not equivalent for m.

א) σ**-additivity**.

$$\mu\Big(\cup_{k=1}^{\infty} A_k\Big) = \sum_{k=1}^{\infty} \mu(A_k);$$

ב) **upper semicontinuity**.

$$A_1 \supset A_2 \supset A_3 \supset ..., \qquad A = \cap_{k=1}^{\infty} A_k, \qquad \Rightarrow \qquad \mu(A) = \lim_{k\to\infty} \mu(A_k);$$

ג) **lower semicontinuity**.

$$A_1 \subset A_2 \subset A_3 \subset ..., \qquad A = \cup_{k=1}^{\infty} A_k, \qquad \Rightarrow \qquad \mu(A) = \lim_{k\to\infty} \mu(A_k);$$

ד) **continuity**.

$$\mu(\lim_{k\to\infty} A_k) = \lim_{k\to\infty} \mu(A_k).$$

Hint. Prove that א) $\iff$ ב), א) $\iff$ ג), ד) $\Rightarrow$ א). Prove that ב), ג) imply

$$\mu(\overline{\lim_{n\to\infty}} A_n) = \mu(\cap_{k=1}^{\infty} \cup_{n=k}^{\infty} A_n) = \lim_{k\to\infty} \mu(\cup_{n=k}^{\infty} A_n) \geq \overline{\lim_{k\to\infty}} \mu(A_k),$$

$$\mu(\underline{\lim_{n\to\infty}} A_n) = \mu(\cup_{k=1}^{\infty} \cap_{n=k}^{\infty} A_n) = \lim_{k\to\infty} \mu(\cap_{n=k}^{\infty} A_n) \leq \underline{\lim_{n\to\infty}} \mu(A_k).$$

This will give you א) $\iff$ ד).

Moreover, consider the following example of a measure, which is not σ-additive. Take a subring S of subsets of $[0,1) \cap \mathbf{Q}$, and define

$$S := \{s_{a,b} := [a,b) \cap [0,1) \cap \mathbf{Q}\}, \qquad m(s_{a,b}) = b - a.$$

Prove that for m, ב) and ג) are true, but א) and ד are not. On the other hand for the extension μ of m, all א), ב), ג), ד) are not true (they are equivalent).

5. **Problem 5.**

Definition 3. A pair (X, d) is called a **metric space**, if X is a **set**, and d is a **distance**. More precisely, d is a nonnegative real function $d(x,y)$ defined for any $x, y \in X$, and satisfying

1) $d(x,y) = 0 \iff x = y$,

2) $d(x,y) = d(y,x)$ (the axiom of symmetry)

3) $d(x,z) \leq d(x,y) + d(y,z)$ (the axiom of triangle).

Let μ be a σ-additive measure on a subring $S \subset P(X)$, and let μ^* be the corresponding outer measure on $P(X)$.

a) We say that $A \sim B$ if $\mu^*(A \triangle B) = 0$. Prove that this is an equivalence relation.

b) Let $\tilde{X}$ be a set of all classes $\tilde{A}$ of equivalence. Prove that $(\tilde{X}, d)$ is a metric space, where $d(\tilde{A}, \tilde{B}) := \mu^*(A \triangle B)$. Here $\tilde{A}, \tilde{B}$ are classes of equivalence containing A, B.

6. **Problem 6.** Is it possible to construct a set $G \subset [0, 1]$ such that

   א) G is dense on $[0, 1]$,

   ב) G has measure (length) zero,

   ג) G is not countable?

   Hint. Consider $G := [0, 1] \setminus E$, where E is a union of sets E_i constructed in the previous Assignment, Problem 5, b)*.