Real Analysis, Math 821.

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Assignment VI.

1. **Problem 1.**

   a) Let $f : \mathbf{R^n} \to \mathbf{R}$ be a continuous function, and let $g_1(x), g_2(x), ..., g_n(x)$ be measurable functions (defined on some space X). Prove that $h(x) := f(g_1(x), g_2(x), ..., g_n(x))$ is measurable.

   Hint. The set $\{(t_1, ..., t_n) \in \mathbf{R^n} : \ f(t_1, ..., t_n) > a\}$ can be written as a union of open sets of the type $(a_{k1}, b_{k1}) \times ... \times (a_{kn}, b_{kn})$, $k = 1, 2, ...$ Then

   $$\{x \in \mathbf{R^n} : \ h(x) > a\} = \cup_{k=1}^{\infty} \cap_{i=1}^{n} \{x \in \mathbf{R} : \ a_{ki} < x < b_{ki}\}.$$

   b) A function $f : X \to \mathbf{C}$, $f(x) = u(x) + iv(x)$, is called measurable if $u(x), v(x)$ are measurable. Prove that $|f(x)|$, and $\arg f(x)$ are measurable.

   c) A vector-function $f : X \to \mathbf{R^n}$ is called measurable if the coordinate functions $(f_1(x), f_2(x), ..., f_n(x))$ of the vector $f(x)$ are measurable with respect to some **fixed** basis in $\mathbf{R^n}$. Prove that this definition does not depend on the choice of the basis.

   d) Let $f : X \to \mathbf{R}$. Describe $a \in \mathbf{R}$ such that measurability of $f(x)^a$ implies the measurability of $f(x)$.

2. **Problem 2.**

   a) Let $f(x)$ be a differentiable function on $[0, 1]$. Prove that $f'(x)$ is measurable.

   b) Let $(f_n(x))_{n=1}^{\infty}$ be a sequence of measurable functions. Prove that $\sup\limits_n f_n(x)$, $\inf\limits_n f_n(x)$ are measurable.

   c) Let $(f_n(x))_{n=1}^{\infty}$ be a sequence of measurable functions. Prove that $\overline{\lim\limits_n} f_n(x)$, $\underline{\lim\limits_n} f_n(x)$ are measurable.

3. **Problem 3.**

   a)* Let $f(x)$ be continuous on $[0, 1]$, and let $N_f(c)$ be a function, (called **the indicator of Banach**), defined as a number (possibly infinite) of solutions of the equation $f(x) = c$. Prove that N_f is measurable.

   Hint. For each $n \in \mathbf{N}$ divide $[0, 1]$ into segments $[(j-1)2^{-n}, j2^{-n}]$, $j = 1, 2, ..., 2^n$, and define $N_n(c)$ as a number of segments $[(j-1)2^{-n}, j2^{-n}]$ containing **at least one** solution $f(x) = c$. Show that functions $N_n(c)$ are measurable, and prove that $N_f(c) = \lim\limits_{n\to\infty} N_n(c)$.

   b) Let $f(x)$ be a one-to-one mapping of $[0, 1]$ onto $[0, 1] \times [0, 1]$ defined as follows: let $x = x_1, x_2, x_3, ...x_n, ... \in [0, 1]$ be binary-irrational (this means that among x_i we have infinitely many 0 and 1). Then, $f(x) := (y_1, y_2)$, where $y_1 = (x_1, x_3, x_5, ...)$, and

$y_2 = (x_2, x_4, x_6, ...)$. For binary-rational numbers choose one way of writing them in terms of 0 and 1, and define f the same way.

Prove that f is Lebesgue measurable and preserves the measure (if $B \subset [0,1] \times [0,1]$ is Lebesgue measurable, then $f^{-1}(B)$ is Lebesgue measurable on $[0,1]$, and $M(B) \equiv m(f^{-1}(B))$, where m is Lebesgue measure on $[0,1]$, and M is Lebesgue measure on $[0,1] \times [0,1]$).

c) Let $x = 0, n_1 n_2 n_3, ...,$ $y = 0, m_1 m_2 m_3, ...,$ decimal decompostions of $x, y \in [0,1]$. Define $f(x,y) = k$ provided $n_k = m_k$ and $n_i \neq m_i$ for $i < k$; $f(x,y) = \infty$ provided $n_k \neq m_k$ for all k. Prove that f is Lebesgue measurable and finite almost everywhere.

4. **Problem 4.**

   a) Let $f_n \to f$ and $f_n \to g$ almost everywhere as $n \to \infty$. Prove that f is equivalent to g.

   b) Let $(r_k)_{k=1}^{\infty}$ be a sequence of rationals from $[0,1]$, and let $r_k = p_k/q_k$ be irreducible fraction. Define $f_k(x) := \exp\{-(p_k - xq_k)^2\}$. Prove that $f_k \to 0$ as $k \to \infty$ in measure, but $\lim\limits_{k\to\infty}$ does not exist at any point $x \in [0,1]$.

   Hint. To show that the pointwise limit does not exist proceed as follows. Take any $x \in [0,1]$ and consider a sequence $(r_{k_n})_{n=1}^{\infty}$ such that $\lim\limits_{n\to\infty} r_{k_n}$ exists and not equal to x. Then show that $f_{k_n}(x) \to 0$. On the other hand, there exists a sequence $(r_{k'_n})_{n=1}^{\infty}$ such that $|p_{k'_n}/q_{k'_n} - x| \leq 1/q_{k'_n}$. Prove that $f_{k'_n}(x) \geq e^{-1}$.

   c) Write out a subsequence $(k_l)_{l=1}^{\infty}$ such that $\lim\limits_{l\to\infty} f_{k_l}(x) = 0$ almost everywhere.

5. **Problem 5*.** We saw in class that

$$D(x) = \lim_{n\to\infty} \lim_{m\to\infty} (\cos(2\pi n! x))^m$$

   where $D(x)$ is the Dirichlet function (taking 0 at irrationals, and 1 at rationals).

   a) Prove that you may not write $D(x) = \lim\limits_{n\to\infty} \varphi_n(x)$ where $\varphi_n(x)$ continuous functions.

   Hint. Let $F_n := \{x \in \mathbf{R} : \varphi_n(x) \leq 1/2\}$. Then $\{x \in \mathbf{R} : D(x) < 1/2\} = \varliminf\limits_{n\to\infty} F_n$. But this contradicts the fact that irrationals may not be written as a countable union of closed sets.

   b) Conclude that it is impossible to define a metric d on the space X of all functions $f : \mathbf{R} \to \mathbf{R}$ such that the convergence in d would be equivalent to the pointwise convergence.

   Hint. Assume the contrary, we have a metric d, such that $f_n(x) \to f(x)$ pointwise is the same as $d(f_n, f) \to 0$ as $n \to \infty$. Let $(g_m(x))_{m=1}^{\infty}$, $(f_{m,n}(x))_{m,n=1}^{\infty}$ be such that $d(f_{m,n}, g_m) \to 0$ as $n \to \infty$, and $d(g, g_m) \to 0$ as $m \to \infty$. But then $\forall m \in \mathbf{N}$ there exists $n(m) \in \mathbf{N}$ such that $d(f_{m,n(m)}, g) \to 0$ (use the diagonal process of Cantor). This contradicts a) (why?).