Real Analysis, Math 821.

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Assignment VII.

1. **Problem 1.**

   a) Compute the Lebesgue integral (with respect to usual Lebesgue measure) over $(0,+\infty)$ from the following functions

   $\aleph$) $f(x) = e^{-[x]}$,

   $\beth$) $f(x) = 1/([x+1][x+2])$,

   $\gimel$) $1/[x]!$.

   Here $[x]$ stands for the integer part of x.

   b) Find the set of values of α, β, such that $f(x) := x^\alpha \sin x^\beta$, $x \in (0,1]$,

   $\aleph$) is Lebesgue integrable,

   $\beth$) is "improper" Riemann integrable, i.e. there exists a limit $\lim\limits_{\epsilon\to 0} \int\limits_\epsilon^1 f(x)dx$.

   c) Compute the Lebesgue integral over $[0,\pi/2]$ from the following functions

   $\aleph$) $f(x) = \sin x$,

   $\beth$) $f(x) = \sin x$ for $x \in \mathbf{Q}$, and $f(x) = \cos x$ for $x \in \mathbf{R}\setminus\mathbf{Q}$,

   $\gimel$) $f(x) = \sin x$ for $\cos x \in \mathbf{Q}$, and $f(x) = \sin^2 x$ for $\cos x \in \mathbf{R}\setminus\mathbf{Q}$.

2. **Problem 2.** a) Let φ be an increasing smooth function on $[a,b]$, and let ψ be its inverse on $[\varphi(a),\varphi(b)]$. Prove that

$$\int\limits_a^b \varphi(x)dx = \int\limits_{\varphi(a)}^{\varphi(b)} y\psi'(y)dy,$$

   where the integral is understood as a limit of Lebesgue integral sums.

   b) Let $f(x) \geq 0$ on $[a,b]$, and let m be Lebesgue measure on $\mathbf{R^2}$. Prove that

$$\int\limits_a^b f(x)dx = m(\{(x,y) : \, a \leq x \leq b, \, 0 \leq y \leq f(x)\}).$$

   c) Let $X : \mu(X) < \infty$, and let $f \geq 0$ on X. Prove that f is Lebesgue integrable on X if and only if

$$\sum_{n=0}^{\infty} 2^n \mu(\{x \in X : \, f(x) \geq 2^n\}) < \infty.$$

3. **Problem 3.** Let $x \in [0,1]$ be written as ternary fraction $x = \sum_{i=1}^{\infty} \alpha_i/3^i$, where $\alpha_i = 0, 1, 2$. Assume also that $k(x)$ is the smallest index in this representation for which $\alpha_k = 1$, (if $\alpha_i \neq 1$ then $k(x) = \infty$). Define

$$c(x) := \sum_{i=1}^{k-1} \frac{\alpha_i}{2^{i+1}} + \frac{1}{2^k}.$$

Observe that $c(x)$ does not depend on the choice of the ternary representation of x. The function $c(x)$ is called the **Cantor ladder**, and it is continuous and monotonic (why?). (Observe that $c(x)$ is constant on every interval complement to the Cantor set, and is taking values $1/2^k$, $3/2^k$, $5/2^k$,...,$(2^k-1)/2^k$ on complemental intervals of the k-th rank).

Compute the following integrals with respect to the measure c (for simplicity I denote it by the same letter), defined on a subring of sets $[a,b) \subset [0,1]$ as $c([a,b)) = c(b) - c(a)$. (It does not really matter here whether we take open, closed, half-open, or half-closed intervals because the function c is continuous).

א) $\int_0^1 x^k dc(x)$,

ב) $\int_0^1 e^x dc(x)$,

ג) $\int_0^1 \sin(\pi x) dc(x)$.

Hint. $c(x/3) = 1/2c(x)$, $c(2/3 + x/3) = 1/2 + 1/2c(x)$. Thus,

$$a_k := \int_0^1 x^k dc(x) = \int_0^{1/3} x^k dc(x) + \int_{2/3}^1 x^k dc(x) = \frac{1}{2 \cdot 3^k}\Big(\int_0^1 y^k dc(y) + \int_0^1 (2+y)^k dc(y)\Big) = ...,$$

$$\Psi(a) := \int_0^1 e^{ax} dc(x) = \int_0^{1/3} e^{ax} dc(x) + \int_{2/3}^1 e^{ax} dc(x) = ...$$

4. **Problem 4.** Prove the **Lebesgue** criterion of the **Riemann** integrability: a bounded f is Riemann integrable on $[0,1]$ iff it is continuous on $[0,1]$ almost everywhere.

Hint. Define $m(x) = \lim_{\delta \to +0} m_\delta(x)$, $M(x) = \lim_{\delta \to +0} M_\delta(x)$, where

$$m_\delta(x) = \inf_{x-\delta<y<x+\delta} f(y), \qquad M_\delta(x) = \sup_{x-\delta<y<x+\delta} f(y).$$

a) Take a sequence of partions of $[0,1]$

$$0 = x_0^1 < x_1^1 < ... < x_{n_1}^1 = 1, \qquad ..., \qquad 0 = x_0^i < x_1^i < ... < x_{n_i}^i = 1, \qquad ...,$$

such that $\lambda_i := \max(x^i_{k+1} - x^i_k) \to 0$, as $i \to \infty$, and define $m^i_k := \inf\limits_{[x^i_{k+1}, x^i_k]} f(x)$. Prove that $\lim\limits_{i\to\infty} \varphi_i(x) = m(x)$ almost everywhere, where

$$\varphi_i(x) = m^i_k \qquad for \qquad x \in (x^i_k, x^i_{k+1}), \qquad \varphi_i(x) = 0 \qquad for \qquad x = x^i_0, x^i_1, ..., x^i_{n_i}.$$

b) Prove that

$$\lim_{i\to\infty} \int_0^1 \varphi_i(x)dx = \int_0^1 m(x)dx,$$

and that the corresponding result is true for M.

c) Let s_i, S_i denote the upper and the lower Darbour sums for the integral of Riemann. Observe that (use the Lebesgue theorem on dominated convergence)

$$\int_0^1 \varphi_i(x)dx = \sum_{k=0}^{n_i-1} \int_{x^i_k}^{x^i_{k+1}} \varphi_i(x)dx = \sum_{k=0}^{n_i-1} m^i_k(x^i_{k+1} - x^i_k) = s_i,$$

$$\int_0^1 \widehat{\varphi}_i(x)dx = \sum_{k=0}^{n_i-1} \int_{x^i_k}^{x^i_{k+1}} \widehat{\varphi}_i(x)dx = \sum_{k=0}^{n_i-1} M^i_k(x^i_{k+1} - x^i_k) = S_i,$$

and conclude that

$$\lim_{i\to\infty}(S_i - s_i) = \int_0^1 (M(x) - m(x))dx.$$

d) Observe that

$$\int_0^1 (M(x) - m(x))dx = 0$$

iff f is Riemann integrable.