

# Real Analysis, Math 821.

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## Assignment VII.

### 1. Problem 1.

a) Compute the Lebesgue integral (with respect to usual Lebesgue measure) over  $(0, +\infty)$  from the following functions

ℵ)  $f(x) = e^{-[x]}$ ,

⊃)  $f(x) = 1/([x + 1][x + 2])$ ,

⊔)  $1/[x]!$ .

Here  $[x]$  stands for the integer part of  $x$ .

b) Find the set of values of  $\alpha, \beta$ , such that  $f(x) := x^\alpha \sin x^\beta$ ,  $x \in (0, 1]$ ,

ℵ) is Lebesgue integrable,

⊃) is “improper” Riemann integrable, i.e. there exists a limit  $\lim_{\epsilon \rightarrow 0} \int_{\epsilon}^1 f(x) dx$ .

c) Compute the Lebesgue integral over  $[0, \pi/2]$  from the following functions

ℵ)  $f(x) = \sin x$ ,

⊃)  $f(x) = \sin x$  for  $x \in \mathbf{Q}$ , and  $f(x) = \cos x$  for  $x \in \mathbf{R} \setminus \mathbf{Q}$ ,

⊔)  $f(x) = \sin x$  for  $\cos x \in \mathbf{Q}$ , and  $f(x) = \sin^2 x$  for  $\cos x \in \mathbf{R} \setminus \mathbf{Q}$ .

2. Problem 2. a) Let  $\varphi$  be an increasing smooth function on  $[a, b]$ , and let  $\psi$  be its inverse on  $[\varphi(a), \varphi(b)]$ . Prove that

$$\int_a^b \varphi(x) dx = \int_{\varphi(a)}^{\varphi(b)} y \psi'(y) dy,$$

where the integral is understood as a limit of Lebesgue integral sums.

b) Let  $f(x) \geq 0$  on  $[a, b]$ , and let  $m$  be Lebesgue measure on  $\mathbf{R}^2$ . Prove that

$$\int_a^b f(x) dx = m(\{(x, y) : a \leq x \leq b, 0 \leq y \leq f(x)\}).$$

c) Let  $X : \mu(X) < \infty$ , and let  $f \geq 0$  on  $X$ . Prove that  $f$  is Lebesgue integrable on  $X$  if and only if

$$\sum_{n=0}^{\infty} 2^n \mu(\{x \in X : f(x) \geq 2^n\}) < \infty.$$

3. **Problem 3.** Let  $x \in [0, 1]$  be written as ternary fraction  $x = \sum_{i=1}^{\infty} \alpha_i/3^i$ , where  $\alpha_i = 0, 1, 2$ . Assume also that  $k(x)$  is the smallest index in this representation for which  $\alpha_k = 1$ , (if  $\alpha_i \neq 1$  then  $k(x) = \infty$ ). Define

$$c(x) := \sum_{i=1}^{k-1} \frac{\alpha_i}{2^{i+1}} + \frac{1}{2^k}.$$

Observe that  $c(x)$  does not depend on the choice of the ternary representation of  $x$ . The function  $c(x)$  is called the **Cantor ladder**, and it is continuous and monotonic (why?). (Observe that  $c(x)$  is constant on every interval complementary to the Cantor set, and is taking values  $1/2^k, 3/2^k, 5/2^k, \dots, (2^k - 1)/2^k$  on complementary intervals of the  $k$ -th rank).

Compute the following integrals with respect to the measure  $c$  (for simplicity I denote it by the same letter), defined on a subring of sets  $[a, b] \subset [0, 1]$  as  $c([a, b]) = c(b) - c(a)$ . (It does not really matter here whether we take open, closed, half-open, or half-closed intervals because the function  $c$  is continuous).

$$\aleph) \int_0^1 x^k dc(x),$$

$$\beth) \int_0^1 e^x dc(x),$$

$$\beth) \int_0^1 \sin(\pi x) dc(x).$$

**Hint.**  $c(x/3) = 1/2c(x)$ ,  $c(2/3 + x/3) = 1/2 + 1/2c(x)$ . Thus,

$$a_k := \int_0^1 x^k dc(x) = \int_0^{1/3} x^k dc(x) + \int_{2/3}^1 x^k dc(x) = \frac{1}{2 \cdot 3^k} \left( \int_0^1 y^k dc(y) + \int_0^1 (2+y)^k dc(y) \right) = \dots,$$

$$\Psi(a) := \int_0^1 e^{ax} dc(x) = \int_0^{1/3} e^{ax} dc(x) + \int_{2/3}^1 e^{ax} dc(x) = \dots$$

4. **Problem 4.** Prove the **Lebesgue** criterion of the **Riemann** integrability: a bounded  $f$  is Riemann integrable on  $[0, 1]$  iff it is continuous on  $[0, 1]$  almost everywhere.

**Hint.** Define  $m(x) = \lim_{\delta \rightarrow +0} m_\delta(x)$ ,  $M(x) = \lim_{\delta \rightarrow +0} M_\delta(x)$ , where

$$m_\delta(x) = \inf_{x-\delta < y < x+\delta} f(y), \quad M_\delta(x) = \sup_{x-\delta < y < x+\delta} f(y).$$

- a) Take a sequence of partitions of  $[0, 1]$

$$0 = x_0^1 < x_1^1 < \dots < x_{n_1}^1 = 1, \quad \dots, \quad 0 = x_0^i < x_1^i < \dots < x_{n_i}^i = 1, \quad \dots,$$

such that  $\lambda_i := \max(x_{k+1}^i - x_k^i) \rightarrow 0$ , as  $i \rightarrow \infty$ , and define  $m_k^i := \inf_{[x_{k+1}^i, x_k^i]} f(x)$ . Prove that  $\lim_{i \rightarrow \infty} \varphi_i(x) = m(x)$  almost everywhere, where

$$\varphi_i(x) = m_k^i \quad \text{for} \quad x \in (x_k^i, x_{k+1}^i), \quad \varphi_i(x) = 0 \quad \text{for} \quad x = x_0^i, x_1^i, \dots, x_{n_i}^i.$$

b) Prove that

$$\lim_{i \rightarrow \infty} \int_0^1 \varphi_i(x) dx = \int_0^1 m(x) dx,$$

and that the corresponding result is true for  $M$ .

c) Let  $s_i, S_i$  denote the upper and the lower Darboux sums for the integral of Riemann. Observe that (use the Lebesgue theorem on dominated convergence)

$$\int_0^1 \varphi_i(x) dx = \sum_{k=0}^{n_i-1} \int_{x_k^i}^{x_{k+1}^i} \varphi_i(x) dx = \sum_{k=0}^{n_i-1} m_k^i (x_{k+1}^i - x_k^i) = s_i,$$

$$\int_0^1 \widehat{\varphi}_i(x) dx = \sum_{k=0}^{n_i-1} \int_{x_k^i}^{x_{k+1}^i} \widehat{\varphi}_i(x) dx = \sum_{k=0}^{n_i-1} M_k^i (x_{k+1}^i - x_k^i) = S_i,$$

and conclude that

$$\lim_{i \rightarrow \infty} (S_i - s_i) = \int_0^1 (M(x) - m(x)) dx.$$

d) Observe that

$$\int_0^1 (M(x) - m(x)) dx = 0$$

iff  $f$  is Riemann integrable.