Real Analysis, Math 821.

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Assignment VIII.

1. **Problem 1.**

   a) Let μ be a measure on X, $f : X \to [0,\infty]$ be measurable, $\int\limits_X f(x)d\mu = c$, where $0 < c < \infty$, and $\alpha > 0$ be a constant. Find

   $$\lim_{n\to\infty} \int\limits_X \log\left(1 + \left(\frac{f(x)}{n}\right)^\alpha\right)^n d\mu.$$

   Hint: If $\alpha \geq 1$, the integrands are dominated by $\alpha f(x)$. If $0 < \alpha < 1$, Fatou's lemma can be applied.

   b) Put $f_n(x) := 1_A(x)$ if n is odd, and $f_n(x) := 1 - 1_A(x)$ if n is even. Show that **strict** inequality can occur in Fatou's lemma.

   c) It is easy to guess the limits of

   $$\int\limits_0^n \left(1 - \frac{x}{n}\right)^n e^{x/2}dx, \qquad \int\limits_0^n \left(1 + \frac{x}{n}\right)^n e^{-2x}dx$$

   as $n \to \infty$. Prove that your guesses are correct.

   d) Does

   $$\lim_{n\to\infty} \frac{1}{n} \int\limits_0^1 \log\left(1 + e^{nf(x)}\right)dx$$

   exist for every real $f \in L([0,1], dx)$? If it exists, what is it?

2. **Problem 2.**

   a) Let $\mu(X) < \infty$, $(f_n)_{n=1}^\infty$ be a sequence of bounded measurable functions on X. Assume also that $f_n \to f$ uniformly as $n \to \infty$. Prove that

   $$\lim_{n\to\infty} \int\limits_X f_n(x)d\mu = \int\limits_X f(x)d\mu,$$

   and show that the hypothesis "$\mu(X) < \infty$" can not be omitted.

   Hint: Take $X = \mathbf{R}$, μ - Lebesgue measure, $f_n(x) = 1_{[-n^2,n^2]}(x)/n$.

Definition. Let μ be a measure on X. We say that a sequence of integrable functions $(f_n)_{n=1}^{\infty}$ converges to integrable f in $L^1(X,\mu)$-sense, if for any $\epsilon > 0$ there exist a natural $N = N(\epsilon)$, such that $\forall n > N$ we have

$$\int\limits_X |f_n(x) - f(x)| d\mu < \epsilon.$$

b) Construct a sequence of integrable on $[0,1]$ functions $(f_n)_{n=1}^{\infty}$ satisfying

$$\lim_{n\to\infty} f_n(x) = 0 \;\; \forall x \in [0,1], \qquad \int\limits_{[0,1]} |f_n(x)| dx \le C \;\; \forall n,$$

but such that the sequence does not converge in $L^1([0,1], dx)$.

Hint: Consider $f_n(x) := n1_{(0,1/n](x)}$.

c) Construct a sequence $(f_k)_{k=1}^{\infty}$ of continuous on $[0,1]$ functions, such that

$$0 \le f_k(x) \le 1, \qquad \lim_{k\to\infty} \int\limits_{[0,1]} f_k(x) dx = 0,$$

but such that the sequence converges for no $x \in [0,1]$.

Hint: Consider $f_k(x)$ defined in Assignment VI, Problem 4, b).

d) Let $(f_n)_{n=1}^{\infty}$ be a sequence of nonnegative integrable functions on X converging to f almost everywhere as $n \to \infty$. Prove that

$$\int\limits_X f_n(x) d\mu \to \int\limits_X f(x) d\mu \qquad \text{as} \qquad n \to \infty$$

implies $f_n \to f$ in $L^1(X,\mu)$ as $n \to \infty$.

Hint: Let $\int\limits_X f(x) d\mu := A$. For any $\epsilon > 0$ ther exists a subset $E_1 \subset X$ of finite measure such that

$$\int\limits_{E_1} f(x) d\mu > A - \epsilon.$$

There exists $\delta = \delta(\epsilon) > 0$, such that

$$\int\limits_E f(x) d\mu < \epsilon \qquad \forall E : \; \mu(E) < \delta.$$

Moreover, there exists $E_2 \subset E_1$ such that $\mu(E_1 \setminus E_2) < \delta$, and f_n converges uniformly on E_2. We have

$$\int\limits_X |f_n(x) - f(x)| d\mu = \int\limits_{E_2} |f_n(x) - f(x)| d\mu + \int\limits_{X\setminus E_2} |f_n(x) - f(x)| d\mu = ...$$

3. **Problem 3.** Let E_i, $i = 1, ..., n$ be measurable subsets of $[0, 1]$ such that every point $x \in [0, 1]$ belongs to at least q sets E_i, $q \leq n$. Prove that there exists i such that $m(E_i) \geq q/n$.

4. **Problem 4.** Let $f(x)$ be bounded on $[0, 1]$. Prove that $\int\limits_0^c f(x)dx = 0 \; \forall c \in [0, 1]$ implies $f(x) = 0$ almost everywhere.

   Hint:

   a) Observe that for any open interval $(a, b) \subset [0, 1]$ we have

   $$\int\limits_0^b f(x)dx - \int\limits_0^a f(x)dx = 0,$$

   and the same is true for any open set.

   b) Prove that the same is true for any closed subset of $[0, 1]$.

   c) Prove that the same is true for any subset $X := \cup_{i=1}^{\infty} F_i \subset [0, 1]$ of type F_σ (a set X is called of type F_σ if it is a union of countably many closed sets). To do this, assume at first that $F_i \subset F_{i+1}$ (otherwise put $F_2 = F_1 \cup F_2$, $F_3 = F_1 \cup F_2 \cup F_3$,..., $F_i = \cup_{k+1}^{i} F_k$). Then observe that

   $$X = F_1 \cup (F_2 \setminus F_1) \cup (F_3 \setminus F_2) \cup ... \cup (F_{i+1} \setminus F_i) \cup ...$$