Real Analysis, Math 821.

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Assignment IX.

1. **Problem 1.**

   a) Let $f(x)$ be a **monotonic** function satisfying

   $$\forall x, y \in \mathbf{R}, \qquad f(x) + f(y) = f(x+y), \qquad f(1) = 1.$$

   Prove that $f(x) = x$.

   Hint: Prove that $f(p/q) = p/q$, then consider two **monotonic** sequences $(r_k)_{k=1}^{\infty}$, $(m_k)_{k=1}^{\infty}$, $r_k, m_k \in \mathbf{Q}$, converging to $x \in \mathbf{R}$, $x > 0$. For $x < 0$, observe that $f(x) = -f(-x)$.

   b) Change the word **monotonic** by **measurable** in a), and answer the same question.

   Hint: Observe that

   $$\int_0^1 f(x+y)dy = \int_0^1 (f(x) + f(y))dy = f(x) + \int_0^1 f(y)dy,$$

   and conclude that f is continuous.

   c)* Is it possible to drop the measurability assumption in b)?

2. **Problem 2.** a) Let $f(x)$, $g(x)$ be increasing functions on the real line. Does it follow that $f(x)g(x)$ is increasing?

   b) Construct the monotonic function on the real line which is not continuous **only** in rational points.

   Hint: Consider

   $$f(x) := \sum_{k \in \mathbf{N}:\, r_k < x} \frac{1}{2^k}, \qquad r_k \in \mathbf{Q}.$$

   To show that $f(x)$ is not continuous in rational points observe that $\forall x > r \in \mathbf{Q}$ we have

   $$f(x) = \sum_{r_k < x} \frac{1}{2^k} = \sum_{r_k < r} \frac{1}{2^k} + \sum_{r \le r_k < x} \frac{1}{2^k}, \qquad \sum_{r \le r_k < x} \frac{1}{2^k} > \frac{1}{2^n},$$

   provided r has n-th place in enumeration of rationals.

   To show that f is continuous in $\mathbf{R} \setminus \mathbf{Q}$, observe that

   $$\sup_{\mathbf{R}} f(x) = 1, \qquad \inf_{\mathbf{R}} f(x) = 0,$$

   hence the sum of jumps may not be bigger than 1.

3. **Problem 3.** a) Let $E \subset [0,1]$, and let $f(x)$ be a **bounded** function on E, satisfying $f(x_1) \leq f(x_2)$ $\forall x_1, x_2 \in E$, $x_1 < x_2$. Is it possible to extend this function in such a way that the extension $\phi(x)$ would be nondecreasing on the whole segment $[0,1]$?

   Hint: Let $x_0 = \inf E$. Put $\phi(x) := \sup\limits_{y<x} f(y)$ for $x \in [0,1] \setminus E$, $x > x_0$, and $\phi(x) := \inf\limits_{E} f(y)$ for $x \in [0,1] \setminus E$, $x \leq x_0$.

   b) Change the word **bounded** by **unbounded** in a), and answer the same question.

   Hint: Look at "ends".

4. **Problem 4.** a) Let $E \subset [0,1]$ be **nowhere dense** closed set of a positive measure. Construct an **increasing** continuously differentiable $f(x)$ on $[0,1]$ such that $f'(x) = 0$ $\forall x \in E$.

   Hint: Take
$$f(x) := \int\limits_0^x \phi(t)dt, \qquad \phi(t) := \inf_{y \in E} |y - t|.$$

   Then observe that ϕ is continuous, and $f'(x) = 0$ $\forall x \in E$. To show that f is increasing take $0 \leq x_1 < x_2 \leq 1$, and observe that there exists an interval $(\alpha, \beta) \subset (x_1, x_2)$, such that $(\alpha, \beta) \cap E = \varnothing$.

   b) Is it possible to drop the assumption **nowhere dense** in a)?

   Hint: There exists $(\alpha, \beta) \subset E$ such that $f'(x) = 0$ $\forall x \in (\alpha, \beta)$.

5. **Problem 5.** Prove that $\forall E \subset [0,1]$, $|E| = 0$, there exists a continuous increasing function $f(x)$ on $[0,1]$ such that $\forall x \in E$ we have $f'(x) = +\infty$.

   Hint:

   a) $\forall n \in \mathbf{N}$ there exists an open bounded set G_n such that $E \subset G_n$, and $m(G_n) < 2^{-n}$.

   b) Put $f_n(x) := m(G_n \cap [0,x])$. Then f_n is increasing, continuous, nonnegative, and satisfying $f_n(x) < 2^{-n}$.

   c) A function $f(x) := \sum\limits_{n=1}^{\infty} f_n(x)$ is also increasing, continuous, nonnegative.

   d) Let $x_0 \in E$. Then $[x_0, x_0 + h] \subset G_n$ for some n fixed, provided $|h|$ is small. Prove that $f_n(x_0 + h) = f_n(x_0) + h$.

   e) Use d) to conclude that $\forall k \in \mathbf{N}$ and $|h|$ small we have
$$\frac{f(x_0 + h) - f(x_0)}{h} \geq \sum_{n=1}^{k} \frac{f_n(x_0 + h) - f_n(x_0)}{h} = k.$$