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FRACTIONAL INTEGRALS AND WAVELET TRANSFORMS

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Abstract

Wavelet type representations of fractional integrals and derivatives are studied in the framework of L^p-spaces. These representations generalize the notion of Marchaud's fractional derivative and are intimately connected with the Calderón reproducing formula. By choosing a relevant "wavelet" measure we give a unified representation of the following basic objects in fractional calculus on the real line: the Riemann-Liouville fractional integrals, the Riesz potentials, the conjugate Riesz potentials, the inverses and linear combinations of these operators.

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Introduction

In the present article we announce some results related to representations of various fractional integrals and derivatives in the form

$$A^\alpha f = \frac{1}{c_\nu(\alpha)} \int\limits_0^\infty \frac{f * \nu_t}{t^{1-\alpha}} dt = \lim_{\substack{\varepsilon\to 0\\ \rho\to\infty}} \frac{1}{c_\nu(\alpha)} \int\limits_\varepsilon^\rho \frac{f * \nu_t}{t^{1-\alpha}} dt, \quad \alpha \in \mathbb{C}, \qquad (0.1)$$

where ν_t is a suitable dilated measure (or distribution), $c_\nu(\alpha)$ is a normalizing factor and the limit is understood in the L^p-norm or in the "almost everywhere"-sense. In the case $\alpha = 0$, A^α coincides with the identity operator or with the

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Hilbert transform (depending on ν) and is connected with Calderón's reproducing formula

$$f=\frac{1}{c_{u,v}}\int_0^\infty \frac{f*u_t*v_t}{t}dt,$$

(see e.g. [1], [4], [6], [7] and references therein). Various fractional integrals (and derivatives) can be written in the form (0.1). For example, by choosing $\nu=\delta_1$, (the Dirac unit mass at the point $x=1$) and $c_\nu(\alpha)=\Gamma(\alpha)$ we obtain the well known Liouville fractional integral

$$A^\alpha f=\frac{1}{\Gamma(\alpha)}\int_0^\infty \frac{f(x-t)}{t^{1-\alpha}}dt=\frac{1}{\Gamma(\alpha)}\int_{-\infty}^x \frac{f(y)dy}{(x-y)^{1-\alpha}}.$$

By putting $\nu=\sum\limits_{j=0}^{\ell}\binom{\ell}{j}(-1)^j\delta_j,\ \ Re\,\alpha>-\ell$, with the unit Dirac masses at the points $j=0,1,\cdots,\ell$, one can readily see that $A^\alpha f$ is the Marchaud fractional derivative of f of order $-\alpha$ (up to a constant factor, cf. [5], [9]). By choosing different ν, this list of examples can be continued.

Our goal is to show that *all* basic operators in fractional calculus can be represented in the form (0.1). We also answer the following question: what classes of measures (or distributions) generate concrete types of fractional integrals and derivatives. Clearly, for $Re\,\alpha\le 0$ the measure ν must enjoy some cancellation properties or, in other words, ν should be a "wavelet measure". Owing to this phenomenon, the representations (0.1) may be called *wavelet type representations of fractional integrals (or derivatives).*

The proofs of the statements presented below can be found in [8].

Notation and Definitions

Below $\mathbb{Z},\mathbb{N},\mathbb{R}$ and $\mathbb{C}$ denote the set of all integers, positive integers, real numbers and complex numbers respectively; [a] is the integer part of $a\in\mathbb{R}$. The notation $C=C(\mathbb{R}),\ L^p=L^p(\mathbb{R})$ for function spaces is standard; $(f(x))_\pm=\max\{\pm f(x),\ 0\}$, $\Phi=\Phi(\mathbb{R})$ is the Lizorkin space of test functions, $\Phi'=\Phi'(\mathbb{R})$ is the dual of Φ (see [3], [9]). For $\omega\in\Phi$ and $f\in\Phi'$ we denote

$$(f,\omega)=\int_{-\infty}^\infty f(x)\overline{\omega(x)}dx,$$

where the integral can be interpreted as the value of the distribution f at the test function $\omega(x)$. We write "$f(x)=g(x)$ in the Φ'-sense" if $(f,\omega)=(g,\omega)$ for all $\omega\in\Phi$. For $g\in\Phi'$ and $t\in\mathbb{R}$ the distribution g_t is defined by $(g_t,\omega)=(g(x),\ \omega(tx))$, $\omega\in\Phi$. In the following $\mathcal{M}$ denotes the set of all complex-valued finite Borel measures μ on the real line $\mathbb{R}$. For $\mu \in \mathcal{M}$ the values $\mu(\{\pm\infty\})$ are assumed to be zero.

$$(H\varphi)(x) = p.v.\frac{1}{\pi i}\int_{-\infty}^{\infty}\frac{\varphi(y)}{x-y}dy$$

is the Hilbert transform of φ. For $\varphi \in L^p,\ 1 \le p \le \infty$ and $\mu \in \mathcal{M}$

$$(\varphi * \mu_t)(x) = \int_{-\infty}^{\infty}\varphi(x-ty)d\mu(y),\ t>0.$$

In the following $\hat{f}$ denotes the Fourier transform of the distribution f; $f^{\vee}$ is the inverse Fourier transform.

Definition 0.1. For $\omega \in \Phi,\ \alpha \in \mathbb{C}$, *the fractional integrals* $I^{\alpha}_{\pm}\omega$, *the fractional derivatives* $\mathcal{D}^{\alpha}_{\pm}\omega$, *the Riesz potential* $I^{\alpha}\omega$, *the conjugate Riesz potential* $I^{\alpha}_{s}\omega$, *the Riesz fractional derivative* $\mathcal{D}^{\alpha}\omega$ *and the conjugate Riesz fractional derivative* $\mathcal{D}^{\alpha}_{s}\omega$ are defined by:

$$(I^{\alpha}_{\pm}\omega)(x) = [(\mp i\xi)^{-\alpha}\hat{\omega}(\xi)]^{\vee}(x), (\mathcal{D}^{\alpha}_{\pm}\omega)(x) = [(\mp i\xi)^{\alpha}\hat{\omega}(\xi)]^{\vee}(x),\ Re\,\alpha \ge 0,$$

$$(I^{\alpha}\omega)(x) = (|\xi|^{-\alpha}\hat{\omega}(\xi))^{\vee}(x),\ (I^{\alpha}_{s}\omega)(x) = (|\xi|^{-\alpha}sgn\xi\,\hat{\omega}(\xi))^{\vee}(x),\ Re\,\alpha \ge 0,$$
$$(\mathcal{D}^{\alpha}\omega)(x) = (|\xi|^{\alpha}\hat{\omega}(\xi))^{\vee}(x),\ (\mathcal{D}^{\alpha}_{s}\omega)(x) = (|\xi|^{\alpha}sgn\xi\,\hat{\omega}(\xi))^{\vee}(x),\ Re\,\alpha \ge 0.$$

where
$$(\mp ix)^{-\alpha} = e^{\alpha\log|x| \mp (\alpha\pi i/2)sgn x}.$$

For the classical representations of these operators see [9].

Definition 0.2. For $f \in \Phi'$ *and* $\alpha \in \mathbb{C}$, *the* Φ'*-distributions* $I^{\alpha}_{\pm}f,\ \mathcal{D}^{\alpha}_{\pm}f,\ I^{\alpha}f,\ \mathcal{D}^{\alpha}f,\ I^{\alpha}_{s}f,\ \mathcal{D}^{\alpha}_{s}f$ are defined by duality as follows:

$$(I^{\alpha}_{\pm}f,\omega) = (f,\overline{I^{\alpha}_{\mp}\bar{\omega}}),\quad (\mathcal{D}^{\alpha}_{\pm}f,\ \omega) = (f,\overline{\mathcal{D}^{\alpha}_{\mp}\bar{\omega}}),\quad (I^{\alpha}f,\omega) = (f,\overline{I^{\alpha}\bar{\omega}}),$$

$$(\mathcal{D}^{\alpha}f,\omega) = (f,\overline{\mathcal{D}^{\alpha}\bar{\omega}}),\quad (I^{\alpha}_{s}f,\omega) = (f,\overline{I^{\alpha}_{s}\bar{\omega}}),\quad (\mathcal{D}^{\alpha}_{s}f,\omega) = (f,\overline{\mathcal{D}^{\alpha}_{s}\bar{\omega}}).$$

1. Basic relations involving Φ'-distributions

Given a Φ'-distribution ν and a complex number α, consider the formal integral

$$A^{\alpha,\nu} = \int_{0}^{\infty}\frac{\nu_t}{t^{1-\alpha}}dt. \tag{1.1}$$

Our goal here is to give sense to this integral and to represent (1.1) as a linear combination of kernels arising in fractional calculus.

For $\alpha \in \mathbb{C}$, consider distributions $h^\alpha_\pm, h^\alpha, h^\alpha_s$, which have the following Fourier transforms in the Φ'-sense (see [9, p. 147], [2, p. 170]):

$$\hat{h}^\alpha_\pm(\xi) = (\mp i\xi)^{-\alpha} = |\xi|^{-\alpha} \exp\Big(\pm \frac{\alpha\pi i}{2} sgn\, \xi\Big), \quad \hat{h}^\alpha(\xi) = |\xi|^{-\alpha}, \quad \hat{h}^\alpha_s(\xi) = sgn\, \xi |\xi|^{-\alpha}.$$

Clearly, $I^\alpha_\pm f = h^\alpha_\pm * f,\ I^\alpha f = h^\alpha * f,\ I^\alpha_s f = h^\alpha_s * f$ for all $\alpha \in \mathbb{C}$ and $f \in \Phi'$. Denote

$$A^{\alpha,\nu}_{\varepsilon,\rho} = \int_\varepsilon^\rho \frac{\nu_t}{t^{1-\alpha}} dt, \quad 0 < \varepsilon < \rho < \infty.$$

Lemma 1.1. *Let $\alpha \in \mathbb{C}$, ν be a Φ'-distribution such that $\hat{\nu}(\xi) \in L_{\rm loc}(\mathbb{R} \setminus \{0\})$. Denote $\hat{\nu}_\pm(\xi) = \dfrac{\hat{\nu}(\xi) \pm \hat{\nu}(-\xi)}{2}$ and let the integrals*

$$a_\pm = \int_0^\infty \frac{\hat{\nu}_\pm(\eta)}{\eta^{1-\alpha}} d\eta$$

exist as the improper ones. Then $A^{\alpha,\nu}_{\varepsilon,\rho} \to a_+ h^\alpha + a_- h^\alpha_s$ as $\varepsilon \to 0,\ \rho \to \infty$ in the Φ'-sense. If $\alpha \notin \mathbb{Z}$, then this limit is also equal to $c_+ h^\alpha_+ + c_- h^\alpha_-$, where $c_\pm = \frac{1}{2}\Big[a_+/\cos(\dfrac{\alpha\pi}{2}) \pm a_-/\sin(\dfrac{\alpha\pi}{2})\Big]$.

Lemma 1.1 shows that the integral $\int_0^\infty (\nu_t * f)(x) dt/t^{1-\alpha}$ may be used for representation of the following operators:

(a) Riesz potentials and their inverses ($a_+ = 1, a_- = 0$);

(b) conjugate Riesz potentials and their inverses ($a_+ = 0, a_- = 1$);

(c) left-sided fractional integrals and derivatives ($c_+ = 1, c_- = 0$);

(d) right-sided fractional integrals and derivatives ($c_+ = 0, c_- = 1$);

(e) integrals and derivatives of integer order.

In a similar way one can represent compositions of mentioned operators with a Hilbert transform and linear combinations (with constant coefficients) of such operators.

2. Wavelet Type Representation of Fractional Derivatives (L^p-theory)

In this section we exhibit natural L^p-analogues of Lemma 1.1 and describe some classes of measures and distributions ν for which these analogues hold.

Theorem 2.1. *Let $Re\, \alpha > 0$, $\nu \in \mathcal{M}$. Assume that*

$$\int_{-\infty}^{\infty} x^j d\nu(x) = 0 \quad \text{for all} \quad j = 0, 1, \cdots, [Re\, \alpha] \quad \text{and} \qquad \int_{|x|>1} |x|^\beta d|\nu|(x) < \infty$$

for some $\beta > Re\,\alpha$. *Let* $f \in L^r$, $1 \le r < \infty$, *and one of the derivatives* $\mathcal{D}^\alpha_\pm f$, $D^\alpha f$, $D^\alpha_s f$ *(in the* Φ'*-sense) belongs to* L^p, $1 < p < \infty$. *Then the limit* $A^\alpha f = \lim\limits_{\varepsilon \to 0} \int_\varepsilon^\infty (\nu_t * f)(x) dt/t^{1+\alpha}$ *exists in the* L^p*-norm and in the a.e. sense, and the following relations hold:*

$$A^\alpha f = \gamma_+ \mathcal{D}^\alpha_+ f + \gamma_- H \mathcal{D}^\alpha_+ f = c_+ \mathcal{D}^\alpha_+ f + c_- \mathcal{D}^\alpha_- f = a_+ \mathcal{D}^\alpha f + a_- \mathcal{D}^\alpha_s f$$

(in the second equality it is assumed $\alpha \notin \mathbb{N}$*) where*

$$\gamma_+ = \int\limits_{-\infty}^{\infty} k^+(x) dx = \begin{cases} \Gamma(-\alpha) \Big[\int\limits_0^\infty x^\alpha d\nu(x) + \cos \alpha\pi \int\limits_{-\infty}^0 |x|^\alpha d\nu(x) \Big], & \alpha \notin \mathbb{N}, \\ \dfrac{(-1)^\ell}{\ell!} \int\limits_{-\infty}^{\infty} x^\ell \log \dfrac{1}{|x|} d\nu(x), & \alpha = \ell \in \mathbb{N}, \end{cases}$$

$$\gamma_- = -\lambda\ \pi i = -\frac{\pi i}{\Gamma(1+\alpha)} \int\limits_{-\infty}^0 |x|^\alpha d\nu(x).$$

$$c_+ = \Gamma(-\alpha) \int\limits_0^\infty x^\alpha d\nu(x), \quad c_- = \Gamma(-\alpha) \int\limits_{-\infty}^0 |x|^\alpha d\nu(x).$$

For $\alpha \notin \mathbb{N}$,

$$a_+ = \Gamma(-\alpha) \cos \frac{\alpha\pi}{2} \int\limits_{-\infty}^\infty |x|^\alpha d\nu(x), \quad a_- = -i\Gamma(-\alpha) \sin \frac{\alpha\pi}{2} \int\limits_{-\infty}^\infty |x|^\alpha sgn\, x\ d\nu(x).$$

For $\alpha = \ell \in \mathbb{N}$,

$$a_+ = \begin{cases} \dfrac{(-1)^{\ell/2}}{\ell!} \int\limits_{-\infty}^\infty x^\ell \log \dfrac{1}{|x|} d\nu(x), & \ell = 2k, k = 1, 2, \cdots, \\ \dfrac{\pi(-1)^{(\ell-1)/2}}{\ell!} \int\limits_{-\infty}^0 x^\ell d\nu(x), & \ell = 2k+1, k = 0, 1, 2, \cdots; \end{cases}$$

$$a_- = \begin{cases} \dfrac{i(-1)^{\ell/2+1}\pi}{\ell!} \int\limits_{-\infty}^0 x^\ell d\nu(x), & \ell = 2k, k = 1, 2, \cdots, \\ \dfrac{i(-1)^{(\ell-1)/2}}{\ell!} \int\limits_{-\infty}^\infty x^\ell \log \dfrac{1}{|x|} d\nu(x), & \ell = 2k+1, k = 0, 1, 2, \cdots. \end{cases}$$

In the case of purely imaginary order operators $\mathcal{D}^\alpha_\pm, \mathcal{D}^\alpha, \mathcal{D}^\alpha_s$ are well determined in Definition 0.1 on functions belonging to Φ. On L^p-functions, $1 < p < \infty$, they are understood as the linear bounded (from L^p into L^p) multiplier operators extended from the dense subset Φ.

Theorem 2.2. *Let* $Re\,\alpha = 0,\ \nu \in \mathcal{M}$,

$$\nu(\mathbb{R}) = 0, \quad \int\limits_{|x|>1} |x|^{\beta} d|\nu|(x) < \infty, \quad \int\limits_{|x|<1} |x|^{-\delta} d|\nu|(x) < \infty$$

for some $\beta > 0$, $\delta \in (0,1)$. *If* $f \in L^p,\ 1 < p < \infty$, *then the limit*

$$A^{\alpha} f = \lim_{\substack{\varepsilon \to 0 \\ \rho \to \infty}} \int\limits_{\varepsilon}^{\rho} (\nu_t * f) dt / t^{1+\alpha}$$

exists in the L^p*-norm and in the a.e. sense, and the following relations hold:*

$$A^{\alpha} f = \gamma_+ \mathcal{D}_+^{\alpha} f + \gamma_- H \mathcal{D}_+^{\alpha} f = c_+ \mathcal{D}_+^{\alpha} f + c_- \mathcal{D}_-^{\alpha} f = a_+ \mathcal{D}^{\alpha} f + a_- \mathcal{D}_s^{\alpha} f$$

(in the second equality it is assumed that $\alpha \neq 0$*). Here* $\gamma_{\pm}$ *are defined by*

$$\gamma_+ = \begin{cases} \Gamma(-\alpha) \Big[\int\limits_0^{\infty} x^{\alpha} d\nu(x) + \cos(\alpha\pi) \int\limits_{-\infty}^{0} |x|^{\alpha} d\nu(x) \Big], & \alpha \neq 0, \\ \int\limits_{-\infty}^{\infty} \log \frac{1}{|x|} d\nu(x), \ \alpha = 0; & \end{cases}$$

$$\gamma_- = -\frac{\pi i}{\Gamma(1+\alpha)} \int\limits_{-\infty}^{0} |x|^{\alpha} d\nu(x),$$

$c_{\pm}$ *are the same as in Theorem 2.1 and* $a_{\pm}$ *can be evaluated by the following formulae:*

$$a_+ = \begin{cases} \Gamma(-\alpha) \cos(\alpha\pi/2) \int\limits_{-\infty}^{\infty} |x|^{\alpha} d\nu(x), & \alpha \neq 0 \\ \int\limits_{-\infty}^{\infty} \log \frac{1}{|x|} d\nu(x), & \alpha = 0. \end{cases}$$

$$a_- = \begin{cases} -i\Gamma(-\alpha) \sin(\alpha\pi/2) \int\limits_{-\infty}^{\infty} |x|^{\alpha} sgn\, x \; d\nu(x), & \alpha \neq 0 \\ \frac{\pi i}{2} \int\limits_{-\infty}^{\infty} sgn\, x \; d\nu(x), & \alpha = 0. \end{cases}$$

Consider some generalizations. Assume that Re $\alpha \geq 1$ and ν belongs to a certain class of Φ'-distributions which satisfies the following condition.

Condition 2.3. *If $\alpha = \ell + \alpha_0, \;\; \ell \in \mathbb{N}, \;\; 0 \leq Re \; \alpha_0 < 1$, then $\nu_0 \overset{(\Phi')}{\equiv} I_+^\ell \nu$ is a finite Borel measure (i.e. $\nu_0 \in \mathcal{M}$) such that*

$$\nu_0(\mathbb{R}) = 0 \quad \textit{and} \quad \int\limits_{|x|>0} |x|^\beta d|\nu_0|(x) < \infty \quad \textit{for some } \beta > Re \; \alpha_0.$$

If Re $\alpha_0 = 0$, we assume additionally, that

$$\int\limits_{|x|<1} |x|^{-\delta} d|\nu_0|(x) < \infty \;\; \textit{for some } \delta \in (0,1).$$

Example 2.4. Let Re $\alpha_0 \neq 0$,

$$\nu(x) = \delta(x-1) - \sum_{j=0}^{\ell} \frac{(-1)^j}{j!} \delta^{(j)}(x),$$

$\delta(x)$ being the Dirac δ-function. It is easy to see that ν satisfies Condition 2.3. Furthermore, for such ν and $f \in \Phi$,

$$\int_0^\infty \frac{\nu_t * f}{t^{1+\alpha}} dt = \int_0^\infty \left[f(x-t) - \sum_{j=0}^{\ell} \frac{(-t)^j}{j!} f^{(j)}(x) \right] \frac{dt}{t^{1+\alpha}}.$$

This coincides with the Gelfand-Shilov regularization of the divergent integral $\int_0^\infty t^{-\alpha-1} f(x-t) dt$ (cf. [2, p. 48]).

Theorem 2.5. *Let $\alpha = \ell + \alpha_0, \;\; \ell \in \mathbb{N}, \;\; Re \; \alpha_0 \in [0,1)$. Assume that ν satisfies Condition 2.3. Given $f \in \Phi'$, let one of the derivatives $\mathcal{D}_\pm^\alpha f$, $\mathcal{D}^\alpha f$, $\mathcal{D}_s^\alpha f$ (in the Φ'-sense) belong to L^p, $1 < p < \infty$. If Re $\alpha \notin \mathbb{N}$, then the truncated integral $\int_\varepsilon^\infty (\nu_t * f)(x) dt/t^{1+\alpha}$ coincides (in the Φ'-sense) with the L^p-function that tends to*

$$\gamma_+ \mathcal{D}_+^\alpha f + \gamma_- H D_+^\alpha f = c_+ \mathcal{D}_+^\alpha f + c_- \mathcal{D}_-^\alpha f = a_+ \mathcal{D}^\alpha f + a_- \mathcal{D}_s^\alpha f$$

in the L^p-norm and in the a.e. sense. Here $\gamma_\pm$, $c_\pm$, $a_\pm$ have the same form as in Theorems 2.1 and 2.3 in which α and ν should be replaced by α_0 and $\nu_0 = I_+^\ell \nu$ respectively. If Re $\alpha \in \mathbb{N}$, then the above statement is valid for the integral $\int_\varepsilon^\rho (\nu_t * f)(x) dt/t^{1+\alpha}$ with $\varepsilon \to 0, \;\; \rho \to \infty$.

3. Wavelet type representation of fractional integrals

In this section we are concerned with integrals which have the form $I^\alpha(\nu, \varphi) = \int_0^\infty (\varphi * \nu_t)(x) \, dt/t^{1-\alpha}$, Re $\alpha > 0$. They represent fractional integrals (and their linear combinations) and can be regarded as solutions to the corresponding differential or, more generally, pseudo-differential equations.

Given a Φ'-distribution f that agrees with a certain locally integrable function, we denote the latter by $[f]_\Phi$.

Theorem 3.1 *Let* $Re\ \alpha > 0$, $\alpha = \ell + \alpha_0$, ℓ *is a nonnegative integer*, $0 \leq Re\ \alpha_0 < 1$. *Assume that* ν *is an integrable function with the following property: the generalized derivative (in the* Φ'*-sense)* $\nu_0 = (d/dx)^\ell \nu$ *belongs to* $\mathcal{M}$ *and satisfies Theorem 2.2. If* $\varphi \in L^p$, $1 \leq p < \infty$ *and one of the* Φ'*-distributions* $I^\alpha_\pm \varphi$, $I^\alpha \varphi$, $I^\alpha_s \varphi$ *agrees with a certain* L^r*-function,* $1 < r < \infty$, *then*

$$\lim_{\substack{\varepsilon \to 0 \\ \rho \to \infty}} \int_\varepsilon^\rho \frac{\varphi * \nu_t}{t^{1-\alpha}} dt = \tilde{\gamma}_+ [I^\alpha_+ \varphi]_\Phi + \tilde{\gamma}_- H [I^\alpha_+ \varphi]_\Phi =$$

$$= \tilde{c}_+ [I^\alpha_+ \varphi]_\Phi + \tilde{c}_- [I^\alpha_- \varphi]_\Phi \quad (\alpha \notin \mathbb{N})$$

$$= \tilde{a}_+ [I^\alpha \varphi]_\Phi + \tilde{a}_- [I^\alpha_s \varphi]_\Phi$$

(the limit being understood in the L^r*-norm and in the a.e. sense) where*

$$\tilde{\gamma}_+ = \begin{cases} \Gamma(\alpha_0) \left[\int_0^\infty x^{-\alpha_0} d\nu_0(x) + \cos(\alpha_0 \pi) \int_{-\infty}^0 x^{-\alpha_0} d\nu_0(x) \right], & \alpha_0 \neq 0, \\ \int_{-\infty}^\infty \log \frac{1}{|x|} d\nu_0(x), & \alpha_0 = 0, \end{cases}$$

$$\tilde{\gamma}_- = -\frac{\pi i}{\Gamma(1-\alpha_0)} \int_{-\infty}^0 |x|^{-\alpha} d\nu_0(x);$$

$$\tilde{c}_+ = \Gamma(\alpha_0) \int_0^\infty x^{-\alpha_0} d\nu_0(x), \qquad \tilde{c}_- = (-1)^\ell \Gamma(\alpha_0) \int_{-\infty}^0 |x|^{-\alpha_0} d\nu_0(x);$$

$$\tilde{a}_+ = (\tilde{c}_+ + \tilde{c}_-) \cos(\alpha\pi/2), \qquad \tilde{a}_- = i(\tilde{c}_+ - \tilde{c}_-) \sin(\alpha\pi/2), \quad \alpha \notin \mathbb{N},$$

$$\tilde{a}_+ = \begin{cases} (-1)^k \int_{-\infty}^\infty \log \frac{1}{|x|} d\nu_0, & \alpha = 2k, \\ (-1)^{k-1} \pi \int_{-\infty}^0 d\nu_0, & \alpha = 2k-1, \quad k \in \mathbb{N}, \end{cases}$$

$$\tilde{a}_- = \begin{cases} (-1)^{k+1} \pi i \int_{-\infty}^0 d\nu_0, & \alpha = 2k, \\ (-1)^{k-1} i \int_{-\infty}^\infty \log \frac{1}{|x|} d\nu_0, & \alpha = 2k-1, \quad k \in \mathbb{N}. \end{cases}$$

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