WEAK-TYPE (p,p) ESTIMATES FOR CERTAIN MAXIMAL OPERATORS

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Abstract. Let $\Phi \in L^q(\mathbf{R}^n)$ have a compact support, and let $f \in L^p(\mathbf{R}^n)$, $p \geq 1$, $\frac{1}{p} + \frac{1}{q} = 1$. We show that the maximal operator $M_\Phi : f \to \sup_{t>0} \Phi_t * f$ has weak-type (p,p) and $\lim_{t\to 0} \Phi_t * f(x)$ exists for a.e. $x \in \mathbf{R}^n$. The result is sharp in the sense that for any $1 \leq s < q$ there exists $\Phi \in L^s(\mathbf{R}^n)$, having a compact support, such that M_Φ is not of weak-type (p,p).

1. Introduction

Let Φ, f be nonnegative functions and let

$$\Phi_t(x) = \frac{1}{t^n}\Phi\left(\frac{x}{t}\right), \qquad \Phi_t * f(x) = \int_{\mathbf{R}^n} \Phi_t(x-y)f(y)dy.$$

Consider the maximal operator

$$M_\Phi : f \to M_\Phi f(x) \equiv \sup_{t>0} \Phi_t * f(x). \tag{1}$$

This operator controls the pointwise convergence of $\Phi_t * f(x)$ as $t \to 0$, and comes up in many problems in harmonic analysis and partial differential equations. See [7] and references contained therein.

A basic unsolved problem is to determine the range of boundedness of the maximal operator M_Φ on the scale of L^p spaces. Our main result is the following.

Theorem. *Let $\Phi \in L^q(\mathbf{R}^n)$, $f \in L^p(\mathbf{R}^n)$, $1 \leq p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$. Suppose also that $supp\,(\Phi) \subseteq B_R(0)$. Then*

$$\lambda^p \left|\{x \in \mathbf{R}^n : M_\Phi f(x) > \lambda\}\right| \leq c_n R^n \|\Phi\|_q^p \|f\|_p^p. \tag{2}$$

Moreover, for any $1 \leq s < q$, there exists $\Phi \in L^s(\mathbf{R}^n)$, having a compact support, such that M_Φ is not of weak-type (p,p).

Operators of this type have been studied before by several authors. For example, when Φ decreases at a sufficiently high rate at infinity, then M_Φ is majorized by the Hardy-Littlewood maximal operator. If for each x, $\Phi(rx)$ is decreasing in r, $0 < r < \infty$, then by the method of rotations, M_Φ is bounded from $L^p(\mathbf{R}^n)$ to itself, $1 < p \leq \infty$. If one strengthens the integrability assumption on Φ by adding a Dini-type condition, then M_Φ is of weak-type $(1,1)$. See [7], page 72. This conclusion also holds in the case $\Phi(x) = \Omega(x/|x|)\chi_{B_1(0)}(x)$, $\Omega\, log^+\Omega \in L^1(S^{n-1})$ (see [4], [5],

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[6]). Here $\chi_{B_1(0)}$ is the characteristic function of the ball with center at the origin of radius 1.

The proof of our main result is based on a multidimensional stopping time argument inspired by a result due to S. Hudson ([1], [2]). This approach is of purely geometric nature and is interesting in its own right. For example, this point of view yields yet another proof of weak-type $(1, 1)$ boundedness of the Hardy-Littlewood maximal operator.

The sharpness of the Theorem follows from the following example (cf. [7], page 81). Let $1 \le p < \infty$ and $1 \le s < q$, $\frac{1}{p} + \frac{1}{q} = 1$. Define

$$\Phi(x) = (1 - |x_1|)^{\epsilon-1}\chi_{[-1,1]^n}(x), \qquad f(x) = x_1^{-\delta}\chi_{[0,1]^n}(x).$$

Then for $1 - 1/s < \epsilon < \delta < 1/p$, we have $\Phi \in L^s$, $f \in L^p$ and

$$M_\Phi f(x) \ge \Phi_{x_1} * f(x) \equiv \infty,$$

provided $x \in \mathbf{R}^n : x_1 > 1, (x_2, x_3, ..., x_n) \in (-x_1, x_1 + 1)^{n-1}$.

Standard arguments (see e.g. [8], Ch. I) imply:

Corollary. *Let $\Phi \in L^q(\mathbf{R}^n)$ have a compact support, and let $f \in L^p(\mathbf{R}^n)$, $1 \le p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$. Then $\lim\limits_{t\to 0} \Phi_t * f(x)$ exists for almost every $x \in \mathbf{R}^n$.*

This paper is organized as follows. In the first section, entitled "Selection Property", we describe a higher dimensional version of the property inspired by ideas of Hudson ([1], [2]). See also ([3]) for a thorough description of related ideas and their applications. In the following section, entitled "Proof of Lemma 2", we estimate a key expression resulting from the linearized version of the maximal operator.

Remark: It is possible that Theorem can be proved by an appropriate extension of known results in multi-linear interpolation. More precisely, by viewing M_Φ as a sub-bi-linear operator, one can show that this operator maps $L^1 \times L^\infty \to L^{1,\infty}$ and $L^\infty \times L^1 \to L^\infty$. If one could prove an appropriate bi-linear version of the Marcienkiewicz interpolation theorem, Theorem would follow. However, our goal is to give a direct geometric argument that exposes the nature of the operator.

2. Selection Property.

Definition. We say that Φ has the selection (p, q) property, $1/p + 1/q = 1$, if for any positive measurable function $t(x)$ defined on a set $D \subset \mathbf{R}^n$, $(0 < |D| < \infty)$ there is a measurable subset $E \subseteq D$ such that

$$|E| \ge a|D|, \tag{3}$$

$$\|S(E, \Phi, t)\|_q \le A\,\|\Phi\|_q\,|D|^{1/q}, \tag{4}$$

where

$$S(E, \Phi, t)(y) \equiv \int_E \Phi\left(\frac{x - y}{t(x)}\right) \frac{dx}{t^n(x)}. \tag{5}$$

Constants here do not depend on $t(x), D, E$.

Lemma 1. *If Φ has the selection (p,q) property, $1 < p < \infty$, then M_Φ satisfies (2).*

Proof. Let $D = \{x \in \mathbf{R}^n : M_\Phi f(x) > \lambda\}$. Without loss of generality we may assume that D is bounded. Then

$$|D| \le \frac{1}{a}|E| \le \frac{1}{a\lambda}\int\limits_E M_\Phi f(x)dx \le \frac{c}{a\lambda}\int\limits_E \frac{dx}{t^n(x)}\int\limits_{\mathbf{R}^n} f(y)\,\Phi\left(\frac{x-y}{t(x)}\right)dy$$

$$= \frac{c}{a\lambda}\int\limits_{\mathbf{R}^n} f(y)dy\int\limits_E \Phi\left(\frac{x-y}{t(x)}\right)\frac{dx}{t^n(x)} \le \frac{c}{a\,\lambda}\|f\|_p\|S(E,\Phi,t)\|_q \le \frac{c\,A}{a\,\lambda}\|f\|_p\|\Phi\|_q\,|D|^{1/q}.$$

□

Lemma 1 reduces the proof of the Theorem to the following estimate.

Lemma 2. *The function $\Phi = \chi_{B_1(0)}$ has the selection $(1,\infty)$ property. More precisely, for any positive measurable function $t(x)$ defined on a set $D \subset \mathbf{R}^n$ of finite measure, there is a subset $\tilde{D} \subseteq D$ satisfying the following properties*

$$|\tilde{D}| > c\,|D|, \tag{6}$$

$$S(\tilde{D},\chi_{B_1(0)},t)(y) \equiv \int\limits_{\tilde{D}\cap B_{t(x)}(y)} \frac{dx}{t^n(x)} \le C. \tag{7}$$

Here $B_r(y) = \{x \in \mathbf{R}^n : |x-y| \le r\}$.

Lemma 3. *Let Φ be as in the Theorem and let $q < \infty$. Then Φ has the selection (p,q) property with $a = c$ from (6) and $A = c\,R^{n/p}$.*

Proof. We take $E = \tilde{D}$ from the previous lemma and observe that (7), together with the Jensen inequality and the Fubini Theorem, yield

$$\|S(E,\Phi,t)\|_q^q \le C^q\,R^{nq}\int\limits_{\mathbf{R}^n} dy \int\limits_{E\cap B_{R\,t(x)}(y)} \Phi^q\left(\frac{x-y}{t(x)}\right)\frac{dx}{R^n\,t^n(x)} =$$

$$= C^q\,R^{nq}\int\limits_E \frac{dx}{R^n\,t^n(x)}\int\limits_{B_{R\,t(x)}(x)} \Phi^q\left(\frac{x-y}{t(x)}\right)dy = C^q\,R^{nq}\,|E|\int\limits_{\mathbf{R}^n}\Phi^q(R\xi)d\xi.$$

By making a substitution $R\xi = z$ in the last integral we obtain the desired result.

□

3. Proof of Lemma 2

We will define the set $\tilde{D}$ as a union of sets $\{\tilde{D}_i\}_{i=0}^\infty$. The procedure described below is a modification of the Calderón-Zygmund stopping time argument. By $l(q)$ we denote the side-length of a dyadic cube q. We construct $\tilde{D}_i$ as follows

$$\tilde{D}_i = \{x \in D \mid x \in q \in Q_{i-1} \text{ and } t(x) > l(q)\}\cup$$

$$\cup\{x \in D \mid x \notin q \in Q_{i-1} \text{ and } t(x) > 2^{-i}\},$$

where Q_{i-1} is a collection of dyadic cubes which we define by induction.

Set $Q_{-1}=\emptyset$ and assume that $Q_0,...,Q_{i-1}$ have already been constructed. Consider the net of dyadic cubes q with $l(q)=2^{-i}$. The construction of Q_i consists of two steps.

Step 1: We choose from the net all those cubes which do not intersect cubes from Q_{i-1} and for which one of the following conditions holds:

$$(a)\quad |q|^{-1}\int\limits_q S(\tilde{D}_i,\chi_{B_1(0)},t)(y)dy>1/2,$$

where

$$S(\tilde{D}_i,\chi_{B_1(0)},t)(y)=\int\limits_{\tilde{D}_i\cap B_{t(x)}(y)}\frac{dx}{t^n(x)},$$

$$(b)\quad \frac{|\tilde{D}_i\cap q|}{|q|}>\frac{1}{2}.$$

Step 2: We add all neighbors from the net to the cubes chosen before ($Q_{i-1}\cup\{$ cubes chosen by step 1 $\}$).

Set $Q_i=Q_{i-1}\cup\{$ cubes chosen by step 1 $\}\cup\{$ cubes chosen by step 2 $\}$. If a cube q satisfies (a) and (b) we say that it is chosen by (a).

We claim that $\cup_{i=0}^{\infty}\tilde{D}_i=\tilde{D}$ is the desired set. First of all

$$\tilde{D}\subset D\subset\bigcup_{q\in Q_i,i=0}^{\infty}q. \tag{8}$$

The first inclusion is obvious. The second one follows from the following argument. Fix any $x\in D$. Assume that $t(x)>2^{-i}$ for some i and x does not belong to any cube from Q_{i-1} (otherwise we are done). Then $x\in\tilde{D}_i$. Since almost all points of $\tilde{D}_i$ are points of density, there is a dyadic cube $q^*\ni x$, $l(q^*)=2^{-j}$, $j\geq i$ and such that $|\tilde{D}_i\cap q^*|/|q^*|>1/2$. Since $\tilde{D}_i\subseteq\tilde{D}_j$, $|\tilde{D}_j\cap q^*|/|q^*|>1/2$. Thus by (b), $q^*\subseteq q\in Q_j$.

So (6) would easily follow from (8) and

$$\sum_{q\in Q_i,i=0}^{\infty}|q|\leq c_n\,|\tilde{D}|. \tag{9}$$

To prove (9), let us divide the system $\{Q_i\}_{i=0}^{\infty}$ into three disjoint subsystems: $K_1=\{$ cubes chosen by condition $(a)\}$, $K_2=\{$ cubes chosen by condition $(b)\}$, $K_3=\{$ cubes chosen by step 2 $\}$. Then it is obvious that

$$\sum_{q\in K_3}|q|\leq 5^n\sum_{q\in K_1,q\in K_2}|q|.$$

Moreover

$$\sum_{q\in K_2}|q|=\sum_{i=0}^{\infty}\sum_{q\in(Q_i\backslash Q_{i-1})\cap K_2}|q|\leq 2\sum_{i=0}^{\infty}\sum_{q\in(Q_i\backslash Q_{i-1})\cap K_2}|\tilde{D}_i\cap q|\leq$$

$$2\sum_{i=0}^{\infty}\sum_{q\in(Q_i\setminus Q_{i-1})\cap K_2}|\tilde{D}\cap q|\le 2|\tilde{D}|.$$

On the other hand

$$\sum_{q\in K_1}|q|=\sum_{i=0}^{\infty}\sum_{q\in(Q_i\setminus Q_{i-1})\cap K_1}|q|\le 2\sum_{i=0}^{\infty}\sum_{q\in(Q_i\setminus Q_{i-1})\cap K_1}\int\limits_q S(\tilde{D}_i,\chi_{B_1(0)},t)(y)dy\le$$

$$2\sum_{i=0}^{\infty}\sum_{q\in(Q_i\setminus Q_{i-1})\cap K_1}\int\limits_q S(\tilde{D},\chi_{B_1(0)},t)(y)dy\le 2\int\limits_{\mathbf{R}^n}S(\tilde{D},\chi_{B_1(0)},t)(y)dy=$$

$$=2\int\limits_{\mathbf{R}^n}\int\limits_{\tilde{D}\cap B_{t(x)}(y)}\frac{dx}{t^n(x)}dy=2\int\limits_{\tilde{D}}\frac{1}{t^n(x)}\int\limits_{B_{t(x)}(x)}dydx\le 2\,c_n\,|\tilde{D}|.$$

The third line follows by Fubini theorem, $B_{t(x)}(x)=\{y\in\mathbf{R}^n:\ |x-y|\le t(x)\}$. So we have

$$\sum_{q\in Q_i,i=0}^{\infty}|q|=\sum_{q\in K_1}|q|+\sum_{q\in K_2}|q|+\sum_{q\in K_3}|q|\le$$

$$\le(5^n+1)(\sum_{q\in K_1}|q|+\sum_{q\in K_2}|q|)\le c_n\,|\tilde{D}|$$

and it proves (6).

It remains to show (7). At first we observe that $S(\tilde{D},\chi_{B_1(0)},t)(y)>1/2$ implies $y\in q\in Q_i$ for some $i\ge 0$. Indeed, fix any y such that $S(\tilde{D},\chi_{B_1(0)},t)(y)>1/2$. Since $\tilde{D}_j\subset\tilde{D}_{j+1}$ we have $S(\tilde{D}_j,\chi_{B_1(0)},t)(y)>1/2$ for sufficiently large j. By the differentiability of integrals, there is a dyadic cube q^*, such that $|q^*|^{-1}\int\limits_{q^*}S(\tilde{D}_j,\chi_{B_1(0)},t)(x)dx$ $>1/2$ and $l(q^*)=2^{-m}$, $m\ge j$. Now $\tilde{D}_j\subset\tilde{D}_m$ implies $|q^*|^{-1}\int\limits_{q^*}S(\tilde{D}_m,\chi_{B_1(0)},t)(x)dx$ $>1/2$ and (a) gives $q^*\subseteq q\in Q_i$, $l(q)=2^{-i}$, $i\le m$.

Now we decompose $S(\tilde{D},\chi_{B_1(0)},t)(y)$ into two parts. The first part will be estimated pointwise, the second one $-$ by mean. Namely

$$S(\tilde{D},\chi_{B_1(0)},t)(y)=\int\limits_{L\cap B_{t(x)}(y)}\frac{dx}{t^n(x)}+\int\limits_{H\cap B_{t(x)}(y)}\frac{dx}{t^n(x)}$$

$$=\ S(L,\chi_{B_1(0)},t)(y)\ +\ S(H,\chi_{B_1(0)},t)(y),$$

where, $L=\tilde{D}\cap B_{100\,l(q)}(y)$ and $H=\tilde{D}\cap(\mathbf{R}^n\setminus B_{100\,l(q)}(y))$. First, we show that

$$S(L,\chi_{B_1(0)},t)(y)<C. \tag{10}$$

Observe that $x\in L\cap B_{t(x)}(y)$ implies $t(x)\ge l(q)/2$. This is obvious if $|x-y|\ge l(q)/2$, since $t(x)>|x-y|$. If $|x-y|<l(q)/2$, then either $x\in q$, or x belongs to the neighbor

$\tilde q$ of q with $l(\tilde q)\ge l(q)/2$ (use Step 2). But

$$x\in L\subset \tilde D \equiv \bigcup_{q\in Q_i\setminus Q_{i-1}, i=0}^{\infty} \{x\in D\cap q\,:\, t(x)>l(q)\},$$

so $t(x)\ge l(\tilde q)$. This gives

$$S(L,\chi_{B_1(0)},t)(y)\le \Big(\frac{l(q)}{2}\Big)^{-n}|B_{100\,l(q)}(y)| \,=\, C,$$

and we get (10). To finish the proof of the lemma, we should show that

$$S(H,\chi_{B_1(0)},t)(y)\le C. \tag{11}$$

To prove (11), it is enough to get the following estimates

$$S(H\setminus H_{i-1},\chi_{B_1(0)},t)(y)\le C, \qquad S(H_{i-1},\chi_{B_1(0)},t)(y)\le C,$$

where $H_{i-1}=\tilde D_{i-1}\cap(\mathbf{R}^n\setminus B_{100\,l(q)}(y))$. Observe that $S(H\setminus H_{i-1},\chi_{B_1(0)},t)(y)=0$. Indeed, for any $x\in\tilde D\setminus\tilde D_{i-1},\ t(x)\le 2^{-i+1}$ we have $(\tilde D\setminus \tilde D_{i-1})\cap(\mathbf{R}^n\setminus B_{100\,l(q)}(y))\cap B_{t(x)}(y)=\emptyset$. It is left to show that

$$S(H_{i-1},\chi_{B_1(0)},t)(y)\le C. \tag{12}$$

We claim that

$$S(H_{i-1},\chi_{B_1(0)},t)(y)\le c\,\frac{1}{|q^*|}\int\limits_{q^*} S(H_{i-1},\chi_{B_1(0)},t)(\xi)\,d\xi, \tag{13}$$

where q^* has the same center as q, and $l(q^*)=3\,l(q)$. Applying Fubini Theorem we see that the left hand side of (13) equals

$$\int\limits_{H_{i-1}\cap B_{t(x)}(y)}\frac{dx}{t^n(x)}\le c\int\limits_{H_{i-1}}\frac{|q^*\cap\, B_{t(x)}(x)|}{|q^*|}\,\frac{dx}{t^n(x)}.$$

But for every $x\in H_{i-1}\cap B_{t(x)}(y)$, we have

$$\frac{|q^*\cap\, B_{t(x)}(x)|}{|q^*|}\,\ge\, c.$$

This gives the inequality claimed in (13).

It remains to show that the right-hand side of (13) is finite. Let $N(q)$ denote the set of dyadic neighbors of q of the sidelength $l(q)$. Then observe that

$$q^*\subset M\equiv N(\text{father of } q)\cup \text{father of } q.$$

Hence there is a cube $\tilde q\in M$ such that

$$\frac{1}{|q^*|}\int\limits_{q^*} S(H_{i-1},\chi_{B_1(0)},t)(\xi)\,d\xi\le c\,\frac{1}{|\tilde q|}\int\limits_{\tilde q} S(H_{i-1},\chi_{B_1(0)},t)(\xi)\,d\xi. \tag{14}$$

We claim that the mean in the right-hand side of (14) is bounded by $1/2$. Indeed, observe that

$$\tilde q\cap p=\emptyset \qquad \forall\, p\in Q_{i-2}\cup\{\text{cubes chosen by step 1 during stage}\, i-1\}.$$

Otherwise, by step 2, q would be covered by some cube chosen at stage $l \le i-1$, which contradicts the choice of q. This means that

$$\frac{1}{|\tilde{q}|}\int\limits_{\tilde{q}} S(H_{i-1}, \chi_{B_1(0)}, t)(\xi)\,d\xi \le \frac{1}{|\tilde{q}|}\int\limits_{\tilde{q}} S(\tilde{D}_{i-1}, \chi_{B_1(0)}, t)(\xi)\,d\xi < \frac{1}{2}$$

by (a) in the construction of Q_i. This completes the proof of Lemma 2 and the Theorem.

□

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