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SINGULAR INTEGRALS GENERATED BY ZONAL MEASURES

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ABSTRACT. We study L^p-mapping properties of the rough singular integral operator $T_\nu f(x) = \int_0^\infty dr/r \int_{\Sigma_{n-1}} f(x - r\theta) d\nu(\theta)$ depending on a finite Borel measure ν on the unit sphere Σ_{n-1} in $\mathbb{R}^n$. It is shown that the conditions $\sup_{|\xi|=1} \int_{\Sigma_{n-1}} \log (1/|\theta \cdot \xi|) d|\nu|(\theta) < \infty$, $\nu(\Sigma_{n-1}) = 0$ imply the L^p-boundedness of T_ν for all $1 < p < \infty$ provided that $n > 2$ and ν is zonal.

1. INTRODUCTION

Let Σ_{n-1} be the unit sphere in $\mathbb{R}^n$, $\Omega \in L^1(\Sigma_{n-1})$, $\int_{\Sigma_{n-1}} \Omega(\theta) d\theta = 0$. Consider the classical Calderón-Zygmund singular integral operator

$$(1.1)\qquad \begin{aligned}(T_\Omega f)(x) &= \lim_{\substack{\epsilon \to 0\\ \rho \to \infty}} (T_\Omega^{\varepsilon,\rho} f)(x) \\ &= \lim_{\substack{\epsilon \to 0\\ \rho \to \infty}} \int_{\varepsilon < |y| < \rho} f(x - y) \frac{\Omega(y/|y|)}{|y|^n} dy,\end{aligned}$$

arising in a variety of problems (we refer the reader to the books [3], [9]–[11] and the survey article [5] for more background information). It is well known (cf. [10], p. 40) that if

$$(1.2)\qquad \sup_{|\xi|=1} \int_{\Sigma_{n-1}} |\Omega(\theta)| \log \frac{1}{|\theta \cdot \xi|} \, d\theta < \infty,$$

then T_Ω is bounded from $L^2(\mathbb{R}^n)$ into itself. In [5] L. Grafakos and A. Stefanov asked the following

Question. Does (1.2) imply the L^p-boundedness of T_Ω for some $p \neq 2$?

In this note we show that (1.2) implies the L^p-boundedness of T_Ω for all $p \in (1, \infty)$ in the case $n > 2$, provided that Ω is zonal (i.e. invariant under all rotations about the x_n-axis). We also consider a generalization of T_Ω with Ω replaced by a

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finite Borel measure ν on Σ_{n-1} (cf. [8])

$$(T_\nu f)(x) = \lim_{\substack{\epsilon\to 0\\ \rho\to\infty}} (T_\nu^{\varepsilon,\rho} f)(x) = \lim_{\substack{\epsilon\to 0\\ \rho\to\infty}} \int_\varepsilon^\rho \frac{dr}{r} \int_{\Sigma_{n-1}} f(x-r\theta)d\nu(\theta). \tag{1.3}$$

Definition. Let $SO(n-1)$ denote the group of rotations leaving the x_n-axis fixed. A finite Borel measure ν on Σ_{n-1} is called *zonal* if $\int_{\Sigma_{n-1}} f(\gamma\vartheta)d\nu(\vartheta) = \int_{\Sigma_{n-1}} f(\vartheta)d\nu(\vartheta)$ for all $\gamma \in SO(n-1)$ and $f \in L^1(\Sigma_{n-1}; d|\nu|)$.

Our main result is

Theorem. *Suppose that* ν *is zonal,* $\nu(\Sigma_{n-1}) = 0$, $n > 2$.

(a) *If*

$$\int_{\Sigma_{n-1}} \log \frac{1}{|\theta_n|\sqrt{1-\theta_n^2}} d|\nu|(\theta) < \infty, \tag{1.4}$$

then the operator T_ν*, initially defined by* (1.3) *on functions* $f \in C_c^\infty(\mathbb{R}^n)$*, extends to a bounded operator from* L^p *into itself for all* $p \in (1,\infty)$.

(b) *Let* $f \in L^p(\mathbb{R}^n), 1 < p < \infty$. *If*

$$\int_{\Sigma_{n-1}} |\theta_n|^{-\beta}(1-\theta_n^2)^{-\beta/2} d|\nu|(\theta) < \infty \quad \textit{for some } \beta \in (0, 1/2),$$

then $\| \sup_{0<\epsilon<\rho<\infty} |T_\nu^{\varepsilon,\rho} f| \,\|_p \le c_p \, \|f\|_p$, $1 < p < \infty$, *and the limit in* (1.3) *exists in the* L^p*-norm and in the a.e. sense.*

The proof of this theorem employs results of D. K. Watson [12], and of J. Duoandikoetxea and J. L. Rubio de Francia [2].

Corollary. *Suppose that* ν *is zonal,* $\nu(\Sigma_{n-1}) = 0$, $n > 2$. *Then* T_ν *extends to a bounded operator from* L^p *into itself for all* $p \in (1,\infty)$*, provided*

$$\sup_{|\xi|=1} \int_{\Sigma_{n-1}} \log \frac{1}{|\theta\cdot\xi|} d|\nu|(\theta) < \infty. \tag{1.5}$$

2. Proof of the Theorem and the Corollary

2.1. **Auxiliary results.** The proofs of the following two lemmas are technical and given in the Appendix.

Lemma A. *Suppose that* Λ *is an* $SO(n-1)$*-invariant subset of* Σ_{n-1} *and* ν *is zonal. Then for* $\beta \in (0, 1/2)$ *and* $n > 2$,

$$\sup_{|\xi|=1} \int_\Lambda |\theta\cdot\xi|^{-\beta} d|\nu|(\theta) \le c \int_\Lambda |\theta_n|^{-\beta}(1-\theta_n^2)^{-\beta/2} d|\nu|(\theta), \qquad c = c(n,\beta). \tag{2.1}$$

Lemma B. *Let* ν *be zonal,* $n > 2$. *Then conditions* (1.4) *and* (1.5) *are equivalent.*

The next result will be used in the proof of the Theorem. By

$$\hat\mu(\xi) = \int_{\mathbb{R}^n} e^{-2\pi i x\cdot\xi} d\mu(x)$$

we denote the Fourier transform of the finite Borel measure μ.

Theorem C ([12], p. 3). *Let $\{\sigma_j\}_{j\in\mathbb{Z}}$ be a sequence of finite Borel measures on $\mathbb{R}^n$, which for integers $m \ge 0$ admit a splitting $\sigma_j = U_j^m + L_j^m$ into Borel measures U_j^m and L_j^m so that*

$$U_j^m \quad \text{and} \quad L_j^m \quad \text{are supported in } \{x : |x| < c\, 2^j\}, \tag{2.2}$$

$$||L_j^m|| \le c, \qquad |\hat{L}_j^m(\xi)| \le \frac{c\, 2^{am}}{(2^j|\xi|)^\alpha} \quad \text{for some} \quad \alpha > 0, \tag{2.3}$$

$$\sup_j \sum_{m=0}^{\infty} ||U_j^m|| \le c. \tag{2.4}$$

Here c and a are nonnegative constants, independent of m and j. If the operator

$$Tf = \sum_{j=-\infty}^{\infty} \sigma_j * f, \qquad f \in C_c^\infty(\mathbb{R}^n),$$

extends to a bounded operator on $L^2(\mathbb{R}^n)$, then T extends to a bounded operator on $L^p(\mathbb{R}^n)$, $1 < p < \infty$.

2.2. **Proof of the main results.** We start with (a). Denote

$$\Gamma_m = \{\theta \in \Sigma_{n-1} : |\theta_n|\sqrt{1-\theta_n^2} < 2^{-m}\}, \qquad \Gamma_m^c = \Sigma_{n-1} \setminus \Gamma_m, \tag{2.5}$$

and set $\sigma_j = U_j^m + L_j^m$, where the measures L_j^m and U_j^m are defined by

$$(L_j^m, g) = c_\nu \int_{2^j}^{2^{j+1}} \frac{dr}{r} \int_{\Gamma_m^c} g(r\theta)d\nu(\theta), \qquad c_\nu = \frac{1}{||\nu|| \log 2}, \tag{2.6}$$

$$(U_j^m, g) = c_\nu \int_{2^j}^{2^{j+1}} \frac{dr}{r} \int_{\Gamma_m} g(r\theta)d\nu(\theta), \tag{2.7}$$

$g \in C_0(\mathbb{R}^n)$. Suppose that $f \in C_c^\infty(\mathbb{R}^n)$. Then the series

$$T_\nu f(x) = \sum_{j\in\mathbb{Z}} (\sigma_j * f)(x)$$

converges for each $x \in \mathbb{R}^n$. By Lemma B and by reasons which are similar to [10, p. 40], T_ν extends to a bounded operator on $L^2(\mathbb{R}^n)$.

Thus, by Theorem C, it suffices to check (2.2)–(2.4). The validity of (2.2) and the first condition in (2.3) is clear. To check the second inequality in (2.3), we note that

$$(L_j^m)^\wedge(\xi) = c_\nu \int_{\Gamma_m^c} d\nu(\theta) \left\{ \int_{2^j}^{2^{j+1}} e^{-2\pi i r\theta\cdot\xi} \frac{dr}{r} \right\}.$$

The integral in brackets is dominated by $\log 2$ and also by $2^{-j}|\theta\cdot\xi|^{-1}$. Hence, it does not exceed $c\,(2^{-j}|\theta\cdot\xi|)^{-\alpha}$ for any $\alpha \in (0,1)$. Let $\alpha = 1/4$, $\xi' = \xi/|\xi|$. By Lemma A,

$$\begin{aligned} |(L_j^m)^\wedge(\xi)| &\le \frac{c}{(2^j|\xi|)^{1/4}} \int_{\Gamma_m^c} \frac{d|\nu|(\theta)}{|\theta\cdot\xi'|^{1/4}} \\ &\le \frac{c}{(2^j|\xi|)^{1/4}} \int_{\Gamma_m^c} \frac{d|\nu|(\theta)}{(|\theta_n|\sqrt{1-\theta_n^2})^{1/4}} \le \frac{c\, 2^{m/4}}{(2^j|\xi|)^{1/4}}. \end{aligned}$$

It remains to check (2.4). By (2.7),

$$||U_j^m|| \leq c_\nu \int_{2^j}^{2^{j+1}} \frac{dr}{r} \int_{\Gamma_m} d|\nu|(\theta) \leq c \int_{\Gamma_m} d|\nu|(\theta).$$

Hence (see (2.5))

$$\sum_{m=0}^{\infty} ||U_j^m|| \leq c \sum_{m=0}^{\infty} \int_{\Gamma_m} d|\nu|(\theta) = c \int_{\Sigma_{n-1}} d|\nu|(\theta) \left[\sum_{m<\log_2(1/|\theta_n|\sqrt{1-\theta_n^2})} 1 \right]$$
$$\leq c \int_{\Sigma_{n-1}} \log \frac{1}{|\theta_n|\sqrt{1-\theta_n^2}} d|\nu|(\theta) < \infty,$$

which gives (2.4). Statement (a) is proved.

In order to prove (b), we use Theorems A, B and E of [2] and follow the scheme from [2, p. 550]. It suffices to show that

$$|\hat{\sigma}_0(\xi)| \ \leq \ c\,|\xi| \quad \text{if} \ \ |\xi| \leq 2, \qquad |\hat{\sigma}_0(\xi)| \ \leq \ c\,|\xi|^{-1/4} \quad \text{if} \ \ |\xi| > 2.$$

The inequality for $|\xi| \leq 2$ is clear because $\nu(\Sigma_{n-1}) = 0$. For $|\xi| > 2$, by Lemma A we obtain

$$|\hat{\sigma}_0(\xi)| \leq c\, \frac{c}{|\xi|^{1/4}} \int_{\Sigma_{n-1}} \frac{d|\nu|(\theta)}{|\theta \cdot \xi'|^{1/4}}$$
$$\leq \frac{c}{|\xi|^{1/4}} \int_{\Sigma_{n-1}} \frac{d|\nu|(\theta)}{(|\theta_n|\sqrt{1-\theta_n^2})^{1/4}} \leq c\,|\xi|^{-1/4}.$$

The Corollary is a consequence of the Theorem and Lemma B.

Appendix

Proposition. *Let* $n > 2$, *and let* Λ *and* ν *be the same as in Lemma A. If* $f \in L^1(\Sigma_{n-1}; d|\nu|)$, *then*

$$\text{(3.1)} \qquad \int_\Lambda f(\vartheta) d\nu(\vartheta) = \frac{1}{\sigma_{n-2}} \int_\Lambda d\nu(\vartheta) \int_{\Sigma_{n-2}} f(\sqrt{1-\vartheta_n^2}\sigma + \vartheta_n e_n) d\sigma,$$

where $\sigma_{n-2} = |\Sigma_{n-2}| = 2\pi^{(n-1)/2}/\Gamma((n-1)/2)$ *and* $d\sigma$ *is the usual Lebesgue measure on* Σ_{n-2}.

Proof. Let $\vartheta = (\sin\theta)\sigma + (\cos\theta)e_n$, $\sigma \in \Sigma_{n-2}$, $\cos\theta = \vartheta_n$. Then

$$\int_\Lambda f(\vartheta) d\nu(\vartheta) = \int_{SO(n-1)} d\gamma \int_\Lambda f(\gamma\vartheta) d\nu(\vartheta)$$
$$= \int_\Lambda d\nu(\vartheta) \int_{SO(n-1)} f((\sin\theta)\gamma\sigma + (\cos\theta)e_n) d\gamma$$
$$= \frac{1}{\sigma_{n-2}} \int_\Lambda d\nu(\vartheta) \int_{\Sigma_{n-2}} f((\sin\theta)\sigma + (\cos\theta)e_n) d\sigma,$$

which gives (3.1). □

Proof of Lemma A. Let $\xi = (\tilde{\xi}, \xi_n)$, $\tilde{\xi} \in \mathbb{R}^{n-1}$. By (3.1),

$$\begin{aligned}
&\int_\Lambda |\theta \cdot \xi|^{-\beta} d|\nu|(\theta) \\
&\quad = \frac{1}{\sigma_{n-2}} \int_\Lambda d|\nu|(\theta) \int_{\Sigma_{n-2}} |\sqrt{1-\theta_n^2}\; \sigma \cdot \tilde{\xi} + \theta_n \xi_n|^{-\beta} d\sigma \\
&\quad = \frac{\sigma_{n-3}}{\sigma_{n-2}} \int_\Lambda A(\xi, \theta) d|\nu|(\theta), \\
&A(\xi, \theta) = \int_{-1}^{1} |t\sqrt{(1-\theta_n^2)(1-\xi_n^2)} + \theta_n \xi_n|^{-\beta} (1-t^2)^{n/2-2} dt.
\end{aligned} \tag{3.2}$$

If $|\theta_n| \geq \sqrt{1-\xi_n^2}$, then $|\xi_n| \geq \sqrt{1-\theta_n^2}$, and by setting $b = \sqrt{(1-\theta_n^2)(1-\xi_n^2)}/|\theta_n \xi_n| \leq 1$, we have

$$\begin{aligned}
A(\xi, \theta) &\leq \int_{-1}^{1} \frac{(1-t^2)^{n/2-2} dt}{(|\theta_n \xi_n| - |t|\sqrt{(1-\theta_n^2)(1-\xi_n^2)})^\beta} \\
&\leq |\theta_n \xi_n|^{-\beta} \int_{-1}^{1} \frac{(1-t^2)^{n/2-2} dt}{(1-b|t|)^\beta} \leq \\
&\leq 2\, |\theta_n|^{-\beta} (1-\theta_n^2)^{-\beta/2} \int_0^1 \frac{(1-t^2)^{n/2-2} dt}{(1-t)^\beta} \\
&= c\, |\theta_n|^{-\beta} (1-\theta_n^2)^{-\beta/2}, \quad c = \text{const.}
\end{aligned}$$

If $|\theta_n| < \sqrt{1-\xi_n^2}$, i.e. $|\xi_n| < \sqrt{1-\theta_n^2}$, we set $a = \dfrac{-\theta_n \xi_n}{\sqrt{(1-\xi_n^2)(1-\theta_n^2)}} \in (-1, 1)$ and get

$$\begin{aligned}
A(\xi, \theta) &\leq [(1-\theta_n^2)(1-\xi_n^2)]^{-\beta/2} \int_{-1}^{1} \frac{(1-t^2)^{n/2-2} dt}{|t-a|^\beta} \\
&\leq |\theta_n|^{-\beta} (1-\theta_n^2)^{-\beta/2} [I(a) + I(-a)], \\
I(a) &= \int_{-1}^{a} \frac{(1-t^2)^{n/2-2} dt}{(a-t)^\beta}.
\end{aligned} \tag{3.3}$$

By the formulas 2.2.6.1 from [7] and 9.102.2 from [4] we obtain

$$\begin{aligned}
I(a) &= \frac{2^{n/2-2} B(n/2-1, 1-\beta)}{(a+1)^{1+\beta-n/2}} F\left(\frac{n}{2}-1, 2-\frac{n}{2}; \frac{n}{2}-\beta; \frac{a+1}{2}\right) \\
&\leq c(n, \beta) < \infty, \quad 0 < \beta < 1/2.
\end{aligned}$$

The same estimate holds for $I(-a)$. □

Proof of Lemma B. Denote

$$R(\xi_n, \theta_n) = \int_{-1}^{1} (1-t^2)^{n/2-2} \log \frac{1}{|t\sqrt{(1-\theta_n^2)(1-\xi_n^2)} + \theta_n \xi_n|} dt.$$

As in (3.2),

$$\begin{aligned}
&\int_{\Sigma_{n-1}} \log \frac{1}{|\theta \cdot \xi|} d|\nu|(\theta) \\
&\quad = \frac{\sigma_{n-3}}{\sigma_{n-2}} \int_{\Sigma_{n-1}} R(\xi_n, \theta_n) d|\nu|(\theta) \overset{\text{def}}{=} \frac{\sigma_{n-3}}{\sigma_{n-2}} K(\xi_n).
\end{aligned} \tag{3.4}$$

Using the same notation as in the proof of Lemma A we have for $|\theta_n| \geq \sqrt{1-\xi_n^2}$:

$$
\begin{aligned}
R(\xi_n, \theta_n) &\leq \int_{-1}^{1} (1-t^2)^{n/2-2} \log \frac{1}{|\theta_n \xi_n|(1-|t|b)} dt \\
&\leq c_1 \log \frac{1}{|\theta_n \xi_n|} + 2 \int_0^1 (1-t^2)^{n/2-2} \log \frac{1}{1-t} dt \\
&\leq c_1 \log \frac{1}{|\theta_n| \sqrt{1-\theta_n^2}} + c_2;
\end{aligned}
$$

for $|\theta_n| < \sqrt{1-\xi_n^2}$:

$$
\begin{aligned}
R(\xi_n, \theta_n) &\leq \log \frac{1}{\sqrt{(1-\theta_n^2)(1-\xi_n^2)}} \left[c_1 + \int_{-1}^{1} (1-t^2)^{n/2-2} \log \frac{1}{|t-a|} dt \right] \\
&< \log \frac{1}{|\theta_n| \sqrt{1-\theta_n^2}} \left[c_1 + c_2 \int_{-1}^{1} \frac{(1-t^2)^{n/2-2}}{|t-a|^{1/4}} dt \right] \\
&\leq c \log \frac{1}{|\theta_n| \sqrt{1-\theta_n^2}},
\end{aligned}
$$

c being independent of a (see the estimate of the integral in (3.3)). Hence (1.4) implies (1.5). Conversely, if (1.5) holds, then (see (3.4)) $K(0) < \infty$ and $K(\pm 1) < \infty$. Since

$$
\begin{aligned}
K(0) = &\int_{\Sigma_{n-1}} \log \frac{1}{\sqrt{1-\theta_n^2}} d|\nu|(\theta) \int_{-1}^{1} (1-t^2)^{n/2-2} dt \\
&+ \int_{\Sigma_{n-1}} d|\nu|(\theta) \int_{-1}^{1} (1-t^2)^{n/2-2} \log \frac{1}{|t|} dt
\end{aligned}
$$

and

$$
K(\pm 1) = \int_{\Sigma_{n-1}} \log \frac{1}{|\theta_n|} d|\nu|(\theta) \int_{-1}^{1} (1-t^2)^{n/2-2} dt,
$$

then

$$
\int_{\Sigma_{n-1}} \log \frac{1}{\sqrt{1-\theta_n^2}} d|\nu|(\theta) < \infty, \qquad \int_{\Sigma_{n-1}} \log \frac{1}{|\theta_n|} d|\nu|(\theta) < \infty,
$$

and (1.4) follows. □

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