SINGULAR INTEGRALS GENERATED BY ZONAL MEASURES

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ABSTRACT. We study $L^p$-mapping properties of the rough singular integral operator $T_\nu f(x) = \int_0^\infty dr/r \int_{\Sigma_{n-1}} f(x - r\theta)d\nu(\theta)$ depending on a finite Borel measure $\nu$ on the unit sphere $\Sigma_{n-1}$ in $\mathbb{R}^n$. It is shown that the conditions $\sup_{|\theta| \leq \rho} \int_{\Sigma_{n-1}} \log (1/|\theta \cdot \xi|)d\nu(\theta) < \infty$, $\nu(\Sigma_{n-1}) = 0$ imply the $L^p$-boundedness of $T_\nu$ for all $1 < p < \infty$ provided that $n > 2$ and $\nu$ is zonal.

1. INTRODUCTION

Let $\Sigma_{n-1}$ be the unit sphere in $\mathbb{R}^n$, $\Omega \in L^1(\Sigma_{n-1})$, $\int_{\Sigma_{n-1}} \Omega(\theta)d\theta = 0$. Consider the classical Calderón-Zygmund singular integral operator

$$(T_\Omega f)(x) = \lim_{\rho \to 0} \int_{\rho} (T_{\Omega;\rho} f)(x)$$

$$(1.1) \quad = \lim_{\rho \to 0} \int_{\rho < |y| < \rho} f(x - y)\frac{\Omega(y/|y|)}{|y|^n} dy,$$

arising in a variety of problems (we refer the reader to the books [3], [9]-[11] and the survey article [5] for more background information). It is well known (cf. [10], p. 40) that if

$$(1.2) \quad \sup_{|\xi| = 1} \int_{\Sigma_{n-1}} |\Omega(\theta)| \log \frac{1}{|\theta \cdot \xi|} d\theta < \infty,$$

then $T_\Omega$ is bounded from $L^2(\mathbb{R}^n)$ into itself. In [5] I. Grafakos and A. Stefanov asked the following

Question. Does (1.2) imply the $L^p$-boundedness of $T_\Omega$ for some $p \neq 2$?

In this note we show that (1.2) implies the $L^p$-boundedness of $T_\Omega$ for all $p \in (1, \infty)$ in the case $n > 2$, provided that $\Omega$ is zonal (i.e. invariant under all rotations about the $x_n$-axis). We also consider a generalization of $T_\Omega$ with $\Omega$ replaced by a

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finite Borel measure $\nu$ on $\Sigma_{n-1}$ (cf. [8])

$$\begin{align*}
(T_{\theta} f)(x) &= \lim_{\epsilon \to 0} (T_{\theta}^\epsilon f)(x) \\
&= \lim_{\epsilon \to 0} \int_{\epsilon}^{\theta} \frac{d\theta'}{\epsilon} \int_{\Sigma_{n-1}} f(x - r\theta') d\nu(\theta).
\end{align*}$$

(1.3)

**Definition.** Let $SO(n-1)$ denote the group of rotations leaving the $x_n$-axis fixed. A finite Borel measure $\nu$ on $\Sigma_{n-1}$ is called *zonal* if $\int_{\Sigma_{n-1}} f(\gamma\theta) d\nu(\theta) = \int_{\Sigma_{n-1}} f(\theta) d\nu(\theta)$ for all $\gamma \in SO(n-1)$ and $f \in L^1(\Sigma_{n-1}; d|\nu|)$.

Our main result is

**Theorem.** Suppose that $\nu$ is zonal, $\nu(\Sigma_{n-1}) = 0$, $n > 2$.

(a) If

$$\int_{\Sigma_{n-1}} \frac{1}{\sqrt{1 - \theta_n^2}} d\nu(\theta) < \infty,$$

then the operator $T_{\theta}$, initially defined by (1.3) on functions $f \in C^\infty_c(\mathbb{R}^n)$, extends to a bounded operator from $L^p$ into itself for all $p \in (1, \infty)$.

(b) Let $f \in L^p(\mathbb{R}^n), 1 < p < \infty$. If

$$\int_{\Sigma_{n-1}} |\theta_n|^{-\beta} (1 - \theta_n^2)^{-\beta/2} d\nu(\theta) < \infty$$

for some $\beta \in (0, 1/2)$, then $\|\sup_{0 < \epsilon < \infty} |T_{\theta}^\epsilon f|\|_p \leq c_p \|f\|_p$, $1 < p < \infty$, and the limit in (1.3) exists in the $L^p$-norm and in the a.e. sense.

The proof of this theorem employs results of D. K. Watson [12], and of J. Duandikotezia and J. L. Rubio de Francia [2].

**Corollary.** Suppose that $\nu$ is zonal, $\nu(\Sigma_{n-1}) = 0$, $n > 2$. Then $T_{\theta}$ extends to a bounded operator from $L^p$ into itself for all $p \in (1, \infty)$, provided

$$\begin{align*}
\sup_{|k| \leq 1} \int_{\Sigma_{n-1}} \frac{1}{|\theta \cdot \xi|} d\nu(\theta) < \infty.
\end{align*}$$

(1.5)

2. **Proof of the Theorem and the Corollary**

2.1. **Auxiliary results.** The proofs of the following two lemmas are technical and given in the Appendix.

**Lemma A.** Suppose that $\Lambda$ is an $SO(n-1)$-invariant subset of $\Sigma_{n-1}$ and $\nu$ is zonal. Then for $\beta \in (0, 1/2)$ and $n > 2$,

$$\begin{align*}
\sup_{|k| \leq 1} \int_{\Lambda} |\theta \cdot \xi|^{-\beta} d\nu(\theta) \\
\leq c \int_{\Lambda} |\theta_n|^{-\beta} (1 - \theta_n^2)^{-\beta/2} d\nu(\theta), \quad c = c(n, \beta).
\end{align*}$$

(2.1)

**Lemma B.** Let $\nu$ be zonal, $n > 2$. Then conditions (1.4) and (1.5) are equivalent.

The next result will be used in the proof of the Theorem. By

$$\hat{\mu}(\xi) = \int_{\mathbb{R}^n} e^{-2\pi i x \cdot \xi} d\mu(x)$$

we denote the Fourier transform of the finite Borel measure $\mu$. 
Theorem C ([12], p. 3). Let $\{\sigma_j\}_{j \in \mathbb{Z}}$ be a sequence of finite Borel measures on $\mathbb{R}^n$, which for integers $m \geq 0$ admit a splitting $\sigma_j = U_j^m + L_j^m$ into Borel measures $U_j^m$ and $L_j^m$ so that

\begin{align}
(2.2) & \quad U_j^m \quad \text{and} \quad L_j^m \quad \text{are supported in} \quad \{x : |x| < 2^j\}, \\
(2.3) & \quad \|L_j^m\| \leq c, \quad |\dot{L}_j^m(\xi)| \leq \frac{c \cdot 2^m}{(2^j|\xi|)^{\alpha}} \quad \text{for some} \quad \alpha > 0, \\
(2.4) & \quad \sup_j \sum_{m=0}^{\infty} \|U_j^m\| \leq c.
\end{align}

Here $c$ and $\alpha$ are nonnegative constants, independent of $m$ and $j$. If the operator

\[ Tf = \sum_{j=-\infty}^{\infty} \sigma_j * f, \quad f \in C_c^\infty(\mathbb{R}^n), \]

extends to a bounded operator on $L^2(\mathbb{R}^n)$, then $T$ extends to a bounded operator on $L^p(\mathbb{R}^n)$, $1 < p < \infty$.

2.2. Proof of the main results. We start with (a). Denote

\[ \Gamma_m = \{\theta \in \Sigma_{n-1} : |\theta_n| \sqrt{1-\theta_n^2} < 2^{-m}\}, \quad \Gamma_m^c = \Sigma_{n-1} \setminus \Gamma_m, \]

and set $\sigma_j = U_j^m + L_j^m$, where the measures $L_j^m$ and $U_j^m$ are defined by

\begin{align}
(2.6) & \quad (L_j^m, g) = c_\nu \int_{2^j}^{2^{j+1}} \frac{dr}{r} \int_{\Gamma_m} g(r\theta) \, d\nu(\theta), \quad c_\nu = \frac{1}{\|\nu\| \log 2}, \\
(2.7) & \quad (U_j^m, g) = c_\nu \int_{2^j}^{2^{j+1}} \frac{dr}{r} \int_{\Gamma_m^c} g(r\theta) \, d\nu(\theta), \quad g \in C_0(\mathbb{R}^n) \quad \text{Suppose that} \quad f \in C_c^\infty(\mathbb{R}^n) \quad \text{Then the series} \quad T_\nu f(x) = \sum_{j \in \mathbb{Z}} (\sigma_j * f)(x) \quad \text{converges for each} \quad x \in \mathbb{R}^n. \quad \text{By Lemma B and by reasons which are similar to [10, p. 40],} \quad T_\nu \quad \text{extends to a bounded operator on} \quad L^2(\mathbb{R}^n). \]

Thus, by Theorem C, it suffices to check (2.2)–(2.4). The validity of (2.2) and the first condition in (2.3) is clear. To check the second inequality in (2.3), we note that

\[ (L_j^m)^\wedge(\xi) = c_\nu \int_{\Gamma_m} d\nu(\theta) \left\{ \int_{2^j}^{2^{j+1}} e^{-2\pi i r\theta} \frac{dr}{r} \right\}. \]

The integral in brackets is dominated by $\log 2$ and also by $2^{-j} |\theta \cdot \xi|^{-1}$, Hence, it does not exceed $c \cdot 2^{-j} |\theta \cdot \xi|^{-\alpha}$ for any $\alpha \in (0, 1)$. Let $\alpha = 1/4$, $\xi' = \xi/|\xi|$. By Lemma A,

\[ |(L_j^m)^\wedge(\xi)| \leq \frac{c}{(2^j|\xi|)^{1/4}} \int_{\Gamma_m} \frac{d|\nu|(|\theta|)}{|\theta \cdot \xi'|^{1/4}} \]

\[ \leq \frac{c}{(2^j|\xi|)^{1/4}} \int_{\Gamma_m} \frac{d|\nu|(|\theta|)}{(\theta_n \sqrt{1-\theta_n^2})^{1/4}} \leq \frac{c \cdot 2^{m/4}}{(2^j|\xi|)^{1/4}}. \]
It remains to check (2.4). By (2.7),
\[ \| U_j \| \leq c \int_{2^j}^{2^{j+1}} \frac{dr}{r} \int_{\Gamma_m} d|\nu|(\theta) \leq c \int_{\Gamma_m} d|\nu|(\theta). \]

Hence (see (2.5))
\[ \sum_{m=0}^{\infty} \| U_j \| \leq c \sum_{m=0}^{\infty} \int_{\Gamma_m} d|\nu|(\theta) = c \int_{\Sigma_{n-1}} d|\nu|(\theta) \left( \sum_{m<\nu\theta_n} \frac{1}{\sqrt{1-\theta_n^2}} \right), \]

which gives (2.4). Statement (a) is proved.

In order to prove (b) we use Theorems A, B and E of [2] and follow the scheme from [2, p. 550]. It suffices to show that
\[ |\hat{\sigma}_0(\xi)| \leq c |\xi| \text{ if } |\xi| \leq 2, \quad |\hat{\sigma}_0(\xi)| \leq c |\xi|^{-1/4} \text{ if } |\xi| > 2. \]

The inequality for $|\xi| \leq 2$ is clear because $\nu(\Sigma_{n-1}) = 0$. For $|\xi| > 2$, by Lemma A we obtain
\[ |\hat{\sigma}_0(\xi)| \leq c \frac{c}{|\xi|^{1/4}} \int_{\Sigma_{n-1}} \frac{d|\nu|(\theta)}{|\theta \cdot \xi|^{1/4}} \leq c |\xi|^{-1/4}, \]

The Corollary is a consequence of the Theorem and Lemma B.

**APPENDIX**

**Proposition.** Let $n > 2$, and let $A$ and $\nu$ be the same as in Lemma A. If $f \in L^1(\Sigma_{n-1}; d|\nu|)$, then
\[ \int_A f(\vartheta) d\nu(\vartheta) = \frac{1}{\sigma_{n-2}} \int_A d\nu(\vartheta) \int_{\Sigma_{n-2}} f(\sqrt{1-\vartheta_n^2} \sigma + \vartheta_n e_n) d\sigma, \]

where $\sigma_{n-2} = |\Sigma_{n-2}| = 2\pi(n-1)/\Gamma((n-1)/2)$ and $d\sigma$ is the usual Lebesgue measure on $\Sigma_{n-2}$.

**Proof.** Let $\vartheta = (\sin \theta) \sigma + (\cos \theta) e_n$, $\sigma \in \Sigma_{n-2}$, $\cos \theta = \vartheta_n$. Then
\[ \int_A f(\vartheta) d\nu(\vartheta) = \int_{SO(n-1)} d\gamma \int_A f(\gamma \vartheta) d\nu(\vartheta) \]

\[ = \int_A d\nu(\vartheta) \int_{SO(n-1)} f((\sin \theta) \gamma \sigma + (\cos \theta) e_n) d\gamma \]

\[ = \frac{1}{\sigma_{n-2}} \int_A d\nu(\vartheta) \int_{\Sigma_{n-2}} f((\sin \theta) \sigma + (\cos \theta) e_n) d\sigma, \]

which gives (3.1).
Proof of Lemma A. Let $\xi = (\tilde{\xi}, \xi_n)$, $\tilde{\xi} \in \mathbb{R}^{n-1}$. By (3.1),

$$
\int_{\Lambda} |\theta \cdot \xi|^{-\beta} d|\nu|(\theta) = \frac{1}{\sigma_{n-2}} \int_{\Lambda} d|\nu|(\theta) \int_{\Sigma_{n-1}} \sqrt{1 - \theta_n^2 \sigma \cdot \tilde{\xi} + \theta_n \xi_n}^{-\beta} d\sigma
$$

(3.2)

$$
= \frac{\sigma_{n-3}}{\sigma_{n-2}} \int_{\Lambda} A(\xi, \theta) d|\nu|(\theta),
$$

$$
A(\xi, \theta) = \int_{-1}^{1} |t \sqrt{(1 - \theta_n^2 (1 - \xi_n^2)) + \theta_n \xi_n}^{-\beta} (1 - t^2)^{n/2 - 2} dt.
$$

If $|\theta_n| \geq \sqrt{1 - \xi_n^2}$, then $|\xi_n| \geq \sqrt{1 - \theta_n^2}$, and by setting $b = (1 - \theta_n^2 (1 - \xi_n^2))/|\theta_n \xi_n| \leq 1$, we have

$$
A(\xi, \theta) \leq \int_{-1}^{1} (|\theta_n \xi_n| - |t| \sqrt{(1 - \theta_n^2) (1 - \xi_n^2)})^{-\beta} (1 - t^2)^{n/2 - 2} dt
$$

$$
\leq |\theta_n \xi_n|^{-\beta} \int_{-1}^{1} (1 - t^2)^{n/2 - 2} dt
$$

$$
\leq 2 |\theta_n|^{-\beta} (1 - \theta_n^2)^{-\beta/2} \int_{0}^{1} (1 - t^2)^{n/2 - 2} dt
$$

$$
= c |\theta_n|^{-\beta} (1 - \theta_n^2)^{-\beta/2}, \quad c = \text{const.}
$$

If $|\theta_n| < \sqrt{1 - \xi_n^2}$, i.e. $|\xi_n| < \sqrt{1 - \theta_n^2}$, we set $a = \frac{-\theta_n \xi_n}{\sqrt{(1 - \xi_n^2)(1 - \theta_n^2)}} \in (-1, 1)$ and get

$$
A(\xi, \theta) \leq [(1 - \theta_n^2)(1 - \xi_n^2)]^{-\beta/2} \int_{-1}^{1} (1 - t^2)^{n/2 - 2} dt
$$

(3.3)

$$
\leq |\theta_n|^{-\beta} (1 - \theta_n^2)^{-\beta/2} [I(a) + I(-a)],
$$

$$
I(a) = \frac{2^{\frac{\beta n/2 - 2}{2}} \beta (n/2 - 1, 1 - \beta)}{(a + 1)^{1+\beta-n/2}} F \left( \frac{n}{2} - 1, 2 - \frac{n}{2}; -\frac{a}{2}; \frac{a + 1}{2} \right).
$$

By the formulas 2.2.6.1 from [7] and 9.10.2.2 from [4] we obtain

$$
I(a) \leq c(n, \beta) < \infty, \quad 0 < \beta < 1/2.
$$

The same estimate holds for $I(-a)$.

Proof of Lemma B. Denote

$$
R(\xi_n, \theta_n) = \int_{-1}^{1} (1 - t^2)^{n/2 - 2} \log \frac{1}{|t \sqrt{(1 - \theta_n^2 (1 - \xi_n^2)) + \theta_n \xi_n}|} dt.
$$

As in (3.2),

$$
\int_{\Sigma_{n-1}} \log \frac{1}{|\theta \cdot \xi|} d|\nu|(\theta)
$$

(3.4)

$$
= \frac{\sigma_{n-3}}{\sigma_{n-2}} \int_{\Sigma_{n-1}} R(\xi_n, \theta_n) d|\nu|(\theta) \leq \frac{\sigma_{n-3}}{\sigma_{n-2}} K(\xi_n).
$$
Using the same notation as in the proof of Lemma A we have for $|\theta_n| \geq \sqrt{1 - \xi_n^2}$:

$$R(\xi_n, \theta_n) \leq \int_{-1}^{1} (1 - t^2)^{\nu/2 - 2} \log \frac{1}{|\theta_n \xi_n| (1 - |t|)} dt$$

$$\leq c_1 \log \frac{1}{|\theta_n \xi_n|} + 2 \int_{-1}^{1} (1 - t^2)^{\nu/2 - 2} \log \frac{1}{1 - t} dt$$

$$\leq c_1 \log \frac{1}{|\theta_n \xi_n|} + c_2$$

for $|\theta_n| < \sqrt{1 - \xi_n^2}$:

$$R(\xi_n, \theta_n) \leq \log \left( 1 + \xi_n^2 \right) \left( 1 - \xi_n^2 \right) \left( c_1 + \int_{-1}^{1} (1 - t^2)^{\nu/2 - 2} \log \frac{1}{|t - a|} dt \right)$$

$$\leq \log \frac{1}{|\theta_n \xi_n| \sqrt{1 - \theta_n^2}} \left[ c_1 + c_2 \int_{-1}^{1} \frac{(1 - t^2)^{\nu/2 - 2}}{|t - a|^{1/4}} dt \right]$$

$$\leq c \log \frac{1}{|\theta_n \xi_n|} \sqrt{1 - \theta_n^2}$$

c being independent of $a$ (see the estimate of the integral in (3.3)). Hence (1.4) implies (1.5). Conversely, if (1.5) holds, then (see (3.4)) $K(0) < \infty$ and $K(\pm 1) < \infty$. Since

$$K(0) = \int_{\Sigma_n} \log \frac{1}{|\theta_n|} d|\nu|(\theta) \int_{-1}^{1} (1 - t^2)^{\nu/2 - 2} dt$$

$$+ \int_{\Sigma_n} d|\nu|(\theta) \int_{-1}^{1} (1 - t^2)^{\nu/2 - 2} \log \frac{1}{|t|} dt$$

and

$$K(\pm 1) = \int_{\Sigma_n} \log \frac{1}{|\theta_n|} d|\nu|(\theta) \int_{-1}^{1} (1 - t^2)^{\nu/2 - 2} dt,$$

then

$$\int_{\Sigma_n} \log \frac{1}{|\theta_n|} d|\nu|(\theta) < \infty, \quad \int_{\Sigma_n} \log \frac{1}{|\theta_n|} d|\nu|(\theta) < \infty,$$

and (1.4) follows. 

\[\square\]

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