1.2.1 Well-Ordering Property of \( \mathbb{N} \): Every nonempty subset of \( \mathbb{N} \) has a least (smallest) element.

1.2.2 Principle of Mathematical Induction:
Let \( S \) be a subset of \( \mathbb{N} \) that possesses the two properties:
1. The number 1 in \( S \).
2. For every \( k \) in \( \mathbb{N} \), if \( k \) in \( S \), then \( k + 1 \) in \( S \).
Then we have \( S = \mathbb{N} \).

* Proof 1.2.1 implies 1.2.2

1.3.8 Theorem The set \( \mathbb{N} \times \mathbb{N} \) is denumerable

1.3.10 Theorem The following statements are equivalent:
(a) \( S \) is a countable set.
(b) There exists a surjection of \( \mathbb{N} \) onto \( S \).
(c) There exists an injection of \( S \) into \( \mathbb{N} \).

1.3.11 Theorem The set \( \mathbb{Q} \) of all rational numbers is denumerable.

1.3.12 Theorem If \( A_m \) is a countable set for each \( m \) in \( \mathbb{N} \), then the union \( A = \bigcup_{m=1}^{\infty} A_m \) is countable.

* Proof of Theorem 1.3.12

1.3.13 Cantor’s Theorem If \( A \) is any set, then there is no surjection of \( A \) onto the power set of \( A \), the Set of all subsets of \( A \). [not proof]

2.1.9 Theorem If \( a \) in \( \mathbb{R} \) is such that \( 0 \leq a < e \) for every \( e > 0 \), then \( a = 0 \).

* Proof of Theorem 2.1.9

Arithmetic-Geometric Mean Inequality: If \( a, b \geq 0 \), then \( \sqrt{ab} \leq \frac{a+b}{2} \); equality holds iff \( ab = 0 \).

*Proof of Arithmetic-Geometric Mean Inequality.

2.2.3 Triangle Inequality If \( a, b \) in \( \mathbb{R} \), then \( |a + b| \leq |a| + |b| \); equality holds iff \( ab \geq 0 \).

*Proof of Triangle Inequality.

2.2.4 Corollary If \( a, b \) in \( \mathbb{R} \), then \( |a| - |b| \leq |a - b| \); equality holds iff \( ab \geq 0 \).

*Proof of Triangle Inequality corollary 2.2.4.

2.3.3 Lemma A number \( u \) is the supremum of a nonempty subset \( S \) of \( \mathbb{R} \) if and only if \( u \) satisfies the conditions:
1. \( s \leq u \) for all \( s \) in \( S \).
2. If \( v < u \), then there exists \( s \) in \( S \) such that \( v < s \)
2a) if \( 0 < e \), then there exists \( s \) in \( S \) such that \( u - e < s \)
2b) if \( n \) in \( \mathbb{N} \), then there exists \( s \) in \( S \) such that \( u - \frac{1}{n} < s \)

*The proof of 2.3.3 and of analogous results for infimums and the discussion above this lemma are important.

2.3.6 The Completeness Property of \( \mathbb{R} \) Every nonempty set of real numbers that has an upper bound also has a supremum in \( \mathbb{R} \). (Statement, use but not proof)
2.4.1 Examples
Suppose that A and B are nonempty subsets of R that satisfy the property: for all a in A and b in B, a ≤ b, Then Sup(A) ≤ Inf(B). (The result and its proof are important.)

2.4.2 Archimedean Property
If x in R, then there exists n in N such that x ≤ n.

2.4.4 Corollary If S = \{\frac{1}{n} | n \in N\} then Inf(S) = 0.

2.4.5 Corollary If t > 0, there exists n in \mathbb{N} such that 0 < 1/n < t.

2.4.6 Corollary If y > 0, there exists n in \mathbb{N} such that n - 1 ≤ y < n.

*Proof of above results 2.4.3 and 2.4.5 the Archimedean Properties of R.

2.4.8 The Density Theorem
If x and y are any real numbers with x < y, then there exists a rational number r in Q such that x < r < y.

2.5.2 Nested Intervals Property of R: Let \( I_n = [a_n, b_n] \) be a sequence of nested, closed intervals of R, with \( I_{n+1} \) contained in \( I_n \) then \( I = \bigcap_{n=1}^{\infty} I_n \) is non-empty.

We actually showed that \( I = [\alpha, \beta] \) where \( \alpha = \text{Sup}(a_n) \) and \( \beta = \text{Inf}(b_n) \), but you do not need to know this proof.

2.5.4 Theorem
The set R of real numbers is not countable

*Proof of 2.5.4

Corollary: the set R\( \setminus \mathbb{Q} \) of irrational numbers is uncountable.

*Proof of above corollary of 2.5.4

3.1.4 Uniqueness of Limits
A sequence in R can have at most one limit.

*Proof of Theorem 3.1.4

3.1.9 Theorem
Let \( X = (x_n) \) be a sequence of real numbers and let m in N. Then the m-tail \( X_m = (x_{m+n}) \) of X converges if and only if X converges. In this case, \( \lim X_m = \lim X \).

*Proof of Theorem 3.1.9

3.2.2 Theorem
A convergent sequence of real numbers is bounded.

*Proof of Theorem 3.2.2

3.2.3 Theorem
(a) Let \( X = (x_n) \) and \( Y = (y_n) \) be sequences of real numbers that converge to x and y, respectively, and let c in R. Then the sequences \( X + Y, X - Y, XY, \) and \( cX \) converge to \( x + y, x - y, xy, \) and \( cx \), respectively.
(b) If \( X = (x_n) \) converges to x and \( Z = (z_n) \) is a sequence of nonzero real numbers that converges to z and if \( z \neq 0 \), then the quotient sequence \( X/Z \) converges to \( x/z \).

The above results are called the Algebra of Limits. Their proofs are illustrative, but are not required.

3.2.4 Theorem
If \( X = (x_n) \) is a convergent sequence of non-negative real numbers, then \( \lim(x_n) \geq 0 \).

*Proof of Theorem 3.2.4
3.2.5 Theorem If \( X = (x_n) \) and \( Y = (y_n) \) are convergent sequences of real numbers and if \( x_n \leq y_n \) for all \( n \) in \( \mathbb{N} \), then \( \lim x_n \leq \lim y_n \).

*Proof of Theorem 3.2.5

3.2.7 Squeeze Theorem Suppose that \( X = (x_n) \), \( Y = (y_n) \), and \( Z = (z_n) \) are sequences of real numbers such that \( x_n \leq y_n \leq z_n \) for all \( n \) in \( \mathbb{N} \), and that \( \lim X = \lim Z \). Then \( Y \) is convergent and \( \lim Y = \lim X = \lim Z \).

The proof of the squeeze theorem is interesting but is not required.

3.2.9 Theorem Let the sequence \( X = (x_n) \) converge to \( x \). Then the sequence \( (|x_n|) \) of absolute values converges to \( |x| \).

*Proof of Theorem 3.2.9

3.2.10 Theorem Let \( X = (x_n) \) be a sequence of non-negative real numbers that converges to \( x \). Then the sequence \( (\sqrt{x_n}) \) of positive square roots converges and its limit is \( \sqrt{x} \).

*Proof of Theorem 3.2.10

3.2.11 Theorem Let \( X = (x_n) \) be a sequence of positive real numbers such that \( L = \lim \left( \frac{2^{n+1}}{x_n} \right) \) exists. If \( L < 1 \), then \( X \) converges and \( \lim X = 0 \).

In the homework we showed if the ratio limit \( L > 1 \), then \( X \) diverges. The case \( L = 1 \) is inconclusive. Both the constant sequence \( (1) \) and the divergent sequence \( (n) \) have ratio limit \( L = 1 \). The ratio limit result is important but the proof is not.

3.3.2 Monotone Convergence Theorem A monotone sequence of real numbers is convergent if and only if it is bounded. Further, if \( X = (x_n) \) is increasing and bounded above, then \( \lim X = \sup(x_n) \) and if \( Y = (y_n) \) is decreasing and bounded below, then \( \lim Y = \inf(y_n) \).

*Theorem 3.3.2 (MCT) and its proof are very important.

3.4.2 Theorem If a sequence \( X = (x_n) \) of real numbers converges to a real number \( x \), then any subsequence \( X' = (x_{n_k}) \) of \( X \) also converges to \( x \).

*Proof of Theorem 3.4.2.

3.4.4 Theorem Let \( X = (x_n) \) be a sequence of real numbers, Then the following are equivalent:

(i) The sequence \( X \) does not converge to \( x \) in \( \mathbb{R} \).

(ii) There exists an \( \epsilon_0 > 0 \) such that for any \( k \) in \( \mathbb{N} \), there exists \( n_k \) in \( \mathbb{N} \) such that \( n_k \geq k \) and \( |x_{n_k} - x| \geq \epsilon_0 \).

(iii) There exists an \( \epsilon_0 > 0 \) and a subsequence \( X' = (x_{n_k}) \) of \( X \) such that for all \( k \) in \( \mathbb{N} \), \( |x_{n_k} - x| \geq \epsilon_0 \).

*Proof of Theorem 3.4.4. The next result is an immediate consequence of 3.4.4 and 3.2.2.

3.4.5 Divergence Criteria If a sequence \( X = (x_n) \) of real numbers has either of the following properties, then \( X \) is divergent.

(i) \( X \) has two convergent subsequences whose limits are not equal.

(ii) \( X \) is unbounded.

3.4.7 Monotone Subsequence Theorem If \( X = (x_n) \) is a sequence of real numbers, then there is a subsequence of \( X \) that is monotone.
3.4.8 The Bolzano-Weierstrass Theorem

A bounded sequence of real numbers has a convergent subsequence.

*Theorem 3.4.8. is an immediate consequence of Theorem 3.4.7 and the MCT. You do not need to know the alternative proof using the Nested Interval Property.

3.4.11 Theorem

If \((x_n)\) is a bounded sequence of real numbers, then the following statements for a real number \(x^*\) are equivalent.

(a) \(x^* = \text{lim sup}(x_n)\).

(b) If \(e > 0\), there are at most a finite number of \(n \in \mathbb{N}\) such that \(x^* + e < x_n\), but an infinite number of \(n \in \mathbb{N}\) such that \(x^* - e < x_n\).

(c) If \(u_m = \sup(x_n : n \geq m)\); then \(x^* = \inf(u_m) = \lim(u_m)\).

(d) If \(S\) is the set of subsequential limits of \((x_n)\) then \(x^* = \sup S\).

* Also, be able to state the analogous characterizations of \(x_* = \text{lim inf}(x_n)\).

3.5.3 Lemma

If \(X = (x_n)\) is a convergent sequence of real numbers, then \(X\) is a Cauchy sequence.

*Proof of Theorem 3.5.3

3.5.4 Lemma

A Cauchy sequence of real numbers is bounded.

*Proof of Theorem 3.5.4

3.5.5 Cauchy Convergence Criterion

A sequence of real numbers is convergent if and only if it is a Cauchy sequence.

*Proof of 3.6.5

3.5.8 Theorem

Every contractive sequence is a Cauchy sequence, and therefore is convergent.

You do not need to know the proof of Theorem 3.5.8 or the estimates on rates of convergence in Theorem 3.5.10.