Calculus II - Test 1 Review - Spring 2016 - Dr. Smithies

Test One is **Friday February 5th**. Unless you have a documented emergency on that date, you may incur a late penalty on a make up test. So, please make every effort to be in the class on time and ready work on this day.

If you have a documented reason why you would like to arrange your test with Student Access Service, you need to contact them. Their website is http://www.kent.edu/sas. Please let me know if you are testing through SAS since I need to email them the test.

Test One will cover sections 4.5, 5.8 and 6.1 - 6.6 of our book. Section 4.5 covers integration by substitution. It is a prerequisite to this class. Section 5.8 is L’Hospital’s Rule. It is also a prerequisite to this class. You need this to calculate the limits in section 6.6. The assigned homework for this material was the problems with solutions in the back of your book from the problem sets:

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These homework problems and the worked examples from class and the book are the best material to review for the test.

**4.5) Substitution method.**

Recall integration is the method for reversing differentiation. The substitution method of integration is the method for reversing the chain rule. It says,

\[
\int f'(g(x))g'(x) \, dx = \int f'(u) \, du \quad \text{where} \quad u = g(x)
\]

This means to integrate a composition, start with a change of variables. Let \( u = g(x) \) be the inside function. Then \( \frac{du}{dx} = g'(x) \) and so \( du = g'(x) \, dx \). Substitution into \( \int f'(g(x))g'(x) \, dx \) give us \( \int f'(u) \, du = f(u) + c \) then substitute the variable \( u \) out and you get the answer \( f(g(x)) + c \).

**5.8) L’Hospital’s Rule.** This says if \( \frac{f(a)}{g(a)} \) has the form \( \frac{0}{0} \) or \( \frac{\infty}{\infty} \) then

\[
\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}.
\]

Here the limit point \( a \) could be either finite or \( \pm \infty \). The usefulness is that the limit of the ratio of the derivatives is often easier to calculate than the limit of the ratio of the functions. The rule only applies to the undetermined forms \( \frac{0}{0} \) or \( \frac{\infty}{\infty} \). However, you can use algebra to express other undetermined forms as one of these two. For example, the undetermined form \( 0 \cdot \infty \) can be rewritten using \( f(x)g(x) = \frac{f(x)}{1/g(x)} \). The undetermined form \( \infty - \infty \) can often be handled by getting a common denominator to create a single fraction. The undermined forms \( 0^0, \infty^0, 1^\infty \) arise from an expression of the form \( \lim_{x \to a} \frac{f(x)^{g(x)}}{g(x)^{f(x)}} \). They have the limit \( e^L \) where \( L = \lim_{x \to a} g(x) \ln(f(x)) \).
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(6.1) Integration by Parts

Integration by parts is the method to reverse the product rule. Integrating both sides of the product rule \( d(uv) = vdu + udv \) gives the integration formula

\[
\int u \, dv = uv - \int v \, du.
\]

You choose the parts \( u \) and \( dv \) and then calculate the parts \( du \) and \( v \). You are hoping the integral of \( vdu \) is easier to handle than the integral of \( udv \). We saw the following problem types in this section:

- **Integrand is clearly a product.** For example \( \int x \sin(x) \, dx \) or \( \int \cos(x) \ln(\sin(x)) \, dx \)
- **Integrand is not a product but substitution does not help.** For example, \( \int \ln(x) \, dx \) or \( \int \cos^{-1}(x) \, dx \).

In this case choose \( u \) to be the integrand and \( dv = dx \).

- **Repeated integration by parts.** For example \( \int x^2 e^x \, dx \) or \( \int (\ln(x))^2 \, dx \). In this case you have to do integration by parts a second time to evaluate the integral of \( vdu \).

- **Integral reoccurs.** For example, \( \int e^x \sin(x) \, dx \) or \( \int \sec^3(x) \, dx \). In this case when you repeat integration by parts you are led to a multiple of the integral you started with. Algebraically solve the equation you get for the desired integral.

Definite integrals are handled in the usual way, by calculating the net change in the antiderivative. That is,

\[
\int_a^b u \, dv = uv\big|_a^b - \int_a^b v \, du.
\]

(6.2) Trig Identities

When an integral involves products and/or powers of trig functions, we use standard trig identities to rewrite the integral. We saw the following types of problems:

- **\( \int \cos^m(x) \sin^n(x) \, dx \).** If \( m \) is odd, isolate the factor \( du = \cos(x) \, dx \) to use with the substitution \( u = \sin(x) \). Express the rest of the integral in terms of \( \sin(x) \) by using the trig identity \( \cos^2(x) = 1 - \sin^2(x) \). The case where \( n \) is odd is handled similarly. If both \( m \) and \( n \) are even, you need to use the half-angle formulas
  \[
  \cos^2(x) = \frac{1}{2}(1 + \cos(2x)) \quad \text{and} \quad \sin^2(x) = \frac{1}{2}(1 - \cos(2x)).
  \]

- **\( \int \tan^m(x) \sec^n(x) \, dx \).** If \( n \) is even, isolate the factor \( du = \sec^2(x) \, dx \) to use with the substitution \( u = \tan(x) \). Express the rest of the integral in terms of \( \tan(x) \) using the trig identity \( 1 + \tan^2(x) = \sec^2(x) \). Similarly if \( m \) is odd and \( n \) is at least 1, you isolate the factor \( du = \sec(x) \tan(x) \, dx \) to use with the substitution \( u = \sec(x) \). Otherwise you may want to express the tangent and secant in terms of sine and cosine. The pair of trig functions \( \cot(x) \) and \( \csc(x) \) work in exactly the same way.

- **The following identities come from the angle addition formulas and are useful when you have a product of different sines or cosines.**
  \[
  \sin(A) \cos(B) = \frac{1}{2} [\sin(A - B) + \sin(A + B)] \\
  \sin(A) \sin(B) = \frac{1}{2} [\cos(A - B) - \cos(A + B)] \\
  \cos(A) \cos(B) = \frac{1}{2} [\cos(A - B) + \cos(A + B)]
  \]

(6.2) Trig Substitutions

In these problems you assume that the variable in the integral is a multiple of a trig function, so that you can use trig identities to simplify the integral. The cases where this is useful are:

- **Expression** \( \sqrt{a^2 - x^2} \) \( x = a \sin(\theta) \)
- **Expression** \( \sqrt{a^2 + x^2} \) \( x = a \tan(\theta) \)
- **Expression** \( \sqrt{x^2 - a^2} \) \( x = a \sec(\theta) \)

Generally, to express your answer in terms of the original variable \( x \), you need to consider the right triangle with angle \( \theta \).
(6.3) Partial Fractions

Partial fractions are used when the integrand has the form of a polynomial divided by a higher degree polynomial. If the degree of the polynomial in the numerator is greater than or equal to the degree of the denominator, use long division to rewrite the quotient. In partial fractions, you express the quotient of two polynomials \( \frac{P(x)}{Q(x)} \) as the sum of fractions whose denominators are the factors of \( Q(x) \) and whose numerators are lower degree polynomials. It is helpful to remember:

- When the denominator \( Q(x) \) has a repeated factor, such as \((ax + b)^r\), you need to use all of the terms \( A_1(ax + b) + A_2(ax + b)^2 + \cdots + A_r(ax + b)^r \) to decompose \( \frac{P(x)}{Q(x)} \).
- The quadratic \( ax^2 + bx + c \) is irreducible (i.e., does not factor) exactly when its discriminant \( b^2 - 4ac \) is negative. In this case, you need to use a term of the form \( \frac{Ax + B}{ax^2 + bx + c} \) in your decomposition.

(6.4) (6.5) Tables, Approximations

It is not always possible to calculate a given integral using our integration methods. One option is to look up the integral on a table; another is to use a program such as Maple or Mathematica. The numerical value of a definite integral \( \int_a^b f(x) \, dx \) can be estimated by methods such as the Midpoint Rule, the Trapezoidal Rule or Simpson’s Rule. These methods work by calculating the first several terms of the Riemann sum which defines the integral, \( \int_a^b f(x) \, dx = \sum_{i=1}^{\infty} f(\hat{x}_i) \Delta x \). We will not test on this material in this class. However, if you would like to explore this topic further, you may propose it as an Honors Project.

(6.6) Improper Integrals

Improper integrals are ones in which the interval of integration is infinite or the integrand has an infinite discontinuity (i.e., division by zero) on a finite interval. In the first type, we evaluate the integral over a infinite interval by evaluating it on a finite interval and taking the limit as the interval extends to infinity. For example,

\[
\int_a^\infty f(x) \, dx = \lim_{t \to \infty} \int_a^t f(x) \, dx.
\]

The other type of improper integral has the form \( \int_a^b f(x) \, dx \) where \( f(x) \) has an infinite discontinuity at some point \( c \) between \( a \) and \( b \). In this case, we use

\[
\int_a^b f(x) \, dx = \int_a^c f(x) \, dx + \int_c^b f(x) \, dx = \lim_{t \to c^-} \int_a^t f(x) \, dx + \lim_{t \to c^+} \int_t^b f(x) \, dx.
\]

The Limit Comparison Theorem says that if \( f(x) \) and \( g(x) \) are continuous functions and \( 0 \leq g(x) \leq f(x) \) for all \( x \geq a \) then

(i) If \( \int_a^\infty f(x) \, dx \) is convergent, then \( \int_a^\infty g(x) \, dx \) is convergent.
(ii) If \( \int_a^\infty g(x) \, dx \) is divergent, then \( \int_a^\infty f(x) \, dx \) is divergent.

This can help you decide is an improper integral will be finite (i.e., convergent).

For a practice test you should do the following worked examples in your book. Be aware that you have to know this material so well that you are fast and accurate in working the problems.

Practice 1: (6.1) 2, 3 (6.2) 11 (6.3) 2 (6.6) 2, 5.

Practice 2: (6.1) 4, 5 (6.2) 12 (6.3) 4 (6.6) 3, 6.

To condense these practice tests to something that can be done in 50 minutes, you probably have to choose one of the two problems listed for sections 6.1, 6.2 and 6.6.