2.1 Using Fundamental Identities

Using the Fundamental Identities

One common application of trigonometric identities is to use given values of trigonometric functions to evaluate other trigonometric functions.

**EXAMPLE 1** Using identities to Evaluate a Function

Use the values $\sec u = -\frac{3}{2}$ and $\tan u > 0$ to find the values of all six trigonometric functions.

**Solution** Using a reciprocal identity, you have

$$\cos u = \frac{1}{\sec u} = \frac{1}{-\frac{3}{2}} = -\frac{2}{3}.$$  

Using a Pythagorean identity, you have

$$\sin^2 u = 1 - \cos^2 u,$$

$$= 1 - \left(-\frac{2}{3}\right)^2 = \frac{5}{9},$$

Pythagorean identity, Substitute $-\frac{2}{3}$ for $\cos u$, Simplify.

Because $\sec u < 0$ and $\tan u > 0$, it follows that $u$ lies in Quadrant III. Moreover, because $\sin u$ is negative when $u$ is in Quadrant III, choose the negative root and obtain $\sin u = -\frac{\sqrt{5}}{3}$. Knowing the values of the sine and cosine enables you to find the values of all six trigonometric functions.

$$\sin u = -\frac{\sqrt{5}}{3}, \quad \cos u = -\frac{2}{3}, \quad \tan u = \frac{\sin u}{\cos u} = \frac{\frac{\sqrt{5}}{3}}{-\frac{2}{3}} = -\frac{\sqrt{5}}{2},$$

$$\csc u = \frac{1}{\sin u} = -\frac{3}{\sqrt{5}} = -\frac{3\sqrt{5}}{5}, \quad \sec u = \frac{1}{\cos u} = -\frac{3}{2},$$

$$\cot u = \frac{1}{\tan u} = \frac{2}{\sqrt{5}} = \frac{2\sqrt{5}}{5}.$$  

**Checkpoint** Audio-video solution in English & Spanish at LarsonPrecalculus.com

Use the values $\tan x = \frac{1}{3}$ and $\cos x < 0$ to find the values of all six trigonometric functions.

**EXAMPLE 2** Simplifying a Trigonometric Expression

Simplify

$$\sin x \cos^2 x - \sin x.$$  

**Solution** First factor out a common monomial factor and then use a fundamental identity.

$$\sin x \cos^2 x - \sin x = \sin x(\cos^2 x - 1)$$

$$= -\sin x(1 - \cos^2 x)$$

$$= -\sin x(\sin^2 x)$$

Pythagorean identity

$$= -\sin^2 x$$

Multiply.

**Checkpoint** Audio-video solution in English & Spanish at LarsonPrecalculus.com

Simplify

$$\cos^2 x \csc x - \csc x.$$
When factoring trigonometric expressions, it is helpful to find a special polynomial factoring form that fits the expression, as shown in Example 3.

**EXAMPLE 3**

**Factoring Trigonometric Expressions**

Factor each expression.

a. \( \sec^2 \theta - 1 \)  
   b. \( 4 \tan^2 \theta + \tan \theta - 3 \)

**Solution**

a. This expression has the form \( u^2 - v^2 \), which is the difference of two squares. It factors as
   \[
   \sec^2 \theta - 1 = (\sec \theta + 1)(\sec \theta - 1).
   \]

b. This expression has the polynomial form \( ax^2 + bx + c \), and it factors as
   \[
   4 \tan^2 \theta + \tan \theta - 3 = (4 \tan \theta - 3)(\tan \theta + 1).
   \]

**Checkpoint**

Factor each expression.

a. \( 1 - \cos^2 \theta \)  
   b. \( 2 \csc^2 \theta - 7 \csc \theta + 6 \)

On occasion, factoring or simplifying can best be done by first rewriting the expression in terms of just one trigonometric function or in terms of sine and cosine only. Examples 4 and 5, respectively, show these strategies.

**EXAMPLE 4**

**Factoring a Trigonometric Expression**

Factor \( \csc^2 x - \cot x - 3 \).

**Solution**

Use the identity \( \csc^2 x = 1 + \cot^2 x \) to rewrite the expression.

\[
\csc^2 x - \cot x - 3 = (1 + \cot^2 x) - \cot x - 3 \quad \text{Pythagorean identity}
\]

\[
= \cot^2 x - \cot x - 2 \quad \text{Combine like terms}
\]

\[
= (\cot x - 2)(\cot x + 1) \quad \text{Factor}
\]

**Checkpoint**

Factor \( \sec^2 x + 3 \tan x + 1 \).

**EXAMPLE 5**

**Simplifying a Trigonometric Expression**

\[
\sin i + \cot i \cos i = \sin i + \left( \frac{\cos i}{\sin i} \right) \cos i \quad \text{Quotient identity}
\]

\[
= \frac{\sin^2 i + \cos^2 i}{\sin i} \quad \text{Add fractions}
\]

\[
= \frac{1}{\sin i} \quad \text{Pythagorean identity}
\]

\[
= \csc i \quad \text{Reciprocal identity}
\]

**Checkpoint**

Simplify \( \csc x - \cos x \cot x \).
2.1 Using Fundamental Identities

**EXAMPLE 1**

Adding Trigonometric Expressions

Perform the addition \( \frac{\sin \theta}{1 + \cos \theta} + \frac{\cos \theta}{\sin \theta} \) and simplify.

**Solution**

\[
\frac{\sin \theta}{1 + \cos \theta} + \frac{\cos \theta}{\sin \theta} = \frac{(\sin \theta)(\sin \theta) + (\cos \theta)(1 + \cos \theta)}{1 + \cos \theta(\sin \theta)}
\]

\[
= \frac{\sin^2 \theta + \cos \theta + \cos \theta}{(1 + \cos \theta)(\sin \theta)}
\]

Multiply.

\[
= \frac{1 + \cos \theta}{(1 + \cos \theta)(\sin \theta)}
\]

Pythagorean identity: \(\sin^2 \theta + \cos^2 \theta = 1\)

\[
= \frac{1}{\sin \theta}
\]

Divide out common factor.

\[
= \csc \theta
\]

Reciprocal identity.

**Checkpoint**

Perform the addition \( \frac{1}{1 + \sin \theta} + \frac{1}{1 - \sin \theta} \) and simplify.

The next two examples involve techniques for rewriting expressions in forms that are used in calculus.

**EXAMPLE 7**

Rewriting a Trigonometric Expression

Rewrite \( \frac{1}{1 + \sin x} \) so that it is not in fractional form.

**Solution**

From the Pythagorean identity

\( \cos^2 x = 1 - \sin^2 x = (1 - \sin x)(1 + \sin x) \)

multiplying both the numerator and the denominator by \( (1 - \sin x) \) will produce a monomial denominator.

\[
\frac{1}{1 + \sin x} = \frac{1}{1 + \sin x} \cdot \frac{1 - \sin x}{1 - \sin x}
\]

Multiply numerator and denominator by \( (1 - \sin x) \).

\[
= \frac{1 - \sin x}{1 - \sin^2 x}
\]

Multiply.

\[
= \frac{1 - \sin x}{\cos^2 x}
\]

Pythagorean identity.

\[
= \frac{1}{\cos^2 x} - \frac{\sin x}{\cos^2 x}
\]

Write as separate fractions.

\[
= \frac{1}{\cos^2 x} - \frac{\sin x}{\cos x \cdot \cos x}
\]

Product of fractions.

\[
= \sec^2 x - \tan x \sec x
\]

Reciprocal and quotient identities.

**Checkpoint**

Rewrite \( \frac{\cos^2 \theta}{1 - \sin \theta} \) so that it is not in fractional form.
Trigonometric Substitution

Use the substitution \( x = 2 \tan \theta, \, 0 < \theta < \pi/2 \), to write
\[
\sqrt{4 + x^2}
\]
as a trigonometric function of \( \theta \).

Solution

Begin by letting \( x = 2 \tan \theta \). Then, you obtain
\[
\sqrt{4 + x^2} = \sqrt{4 + (2 \tan \theta)^2} = \sqrt{4 + 4 \tan^2 \theta} = \sqrt{4(1 + \tan^2 \theta)} = \sqrt{4 \sec^2 \theta} = 2 \sec \theta.
\]

\( \checkmark \) Checkpoint

Audio-video solution in English & Spanish at LarsonPrecalculus.com.

Use the substitution \( x = 3 \sin \theta, \, 0 < \theta < \pi/2 \), to write
\[
\sqrt{9 - x^2}
\]
as a trigonometric function of \( \theta \).

The figure below shows the right triangle illustration of the trigonometric substitution \( x = 2 \tan \theta \) in Example 8.

![Right Triangle Illustration]

Angle whose tangent is \( x/2 \)

Use this triangle to check the solution of Example 8, as follows. For \( 0 < \theta < \pi/2 \), you have

\[
\text{opp} = x, \quad \text{adj} = 2, \quad \text{hyp} = \sqrt{4 + x^2}.
\]

With these expressions, you can write

\[
\sec \theta = \frac{\text{hyp}}{\text{adj}} = \frac{\sqrt{4 + x^2}}{2}.
\]

So, \( 2 \sec \theta = \sqrt{4 + x^2} \), and the solution checks.

Summarize (Section 2.1)

1. State the fundamental trigonometric identities (page 210).
2. Explain how to use the fundamental trigonometric identities to evaluate trigonometric functions, simplify trigonometric expressions, and rewrite trigonometric expressions (pages 211–214). For examples of these concepts, see Examples 1–8.
2.1 Exercises

Vocabulary: Fill in the blank to complete the trigonometric identity.

1. \( \frac{\sin u}{\cos u} = \) ________
2. \( \frac{1}{\csc u} = \) ________
3. \( \frac{1}{\tan u} = \) ________
4. \( \sec \left( \frac{\pi}{2} - u \right) = \) ________
5. \( 1 + \) ________ = \( \csc^2 u \)
6. \( \cot(-u) = \) ________

Skills and Applications

Using identities to evaluate a function. In Exercises 7–14, use the given values to find the values (if possible) of all six trigonometric functions.

7. \( \sin x = \frac{1}{2} \), \( \cos x = \frac{\sqrt{3}}{2} \)
8. \( \csc \theta = \frac{5}{3} \), \( \tan \theta = \frac{7}{24} \)
9. \( \cos \left( \frac{\pi}{2} - x \right) = \frac{3}{5} \), \( \cos x = \frac{4}{5} \)
10. \( \sin(-x) = -\frac{1}{3} \), \( \tan x = -\frac{\sqrt{2}}{4} \)
11. \( \sec x = 4 \), \( \sin x > 0 \)
12. \( \csc \theta = -5 \), \( \cos \theta < 0 \)
13. \( \sin \theta = -1 \), \( \cot \theta = 0 \)
14. \( \tan \theta \) is undefined. \( \sin \theta > 0 \)

Matching Trigonometric Expressions. In Exercises 15–20, match the trigonometric expression with one of the following.

(a) \( \csc x \)  (b) \( -1 \)  (c) \( 1 \)
(d) \( \sin x \tan x \)  (e) \( \sec^2 x \)  (f) \( \sec^2 x + \tan^2 x \)
15. \( \sec x \cos x \)
16. \( \cot^2 x - \csc^2 x \)
17. \( \sec^2 x - \tan^2 x \)
18. \( \cot x \sec x \)
19. \( \frac{\sec^2 x - 1}{\sin^2 x} \)
20. \( \frac{\cos^2(\pi/2) - x}{\cos x} \)

Factoring a Trigonometric Expression. In Exercises 21–28, factor the expression and use the fundamental identities to simplify. There is more than one correct form of each answer.

21. \( \tan^2 x - \sin^2 x \tan^2 x \)
22. \( \sin^2 x \sec^2 x - \sin^2 x \)
23. \( \sec^2 x - 1 \)
24. \( \frac{\cos x - 2}{\cos x - 4} \)
25. \( 1 - 2 \cos^2 x + \cos^2 x \)
26. \( \sec^2 x - \tan^2 x \)
27. \( \cot x + \cot^2 x - \cot x + 1 \)
28. \( \sec^3 x - \sec^2 x - \sec x + 1 \)

Factoring a Trigonometric Expression. In Exercises 29–32, factor the trigonometric expression. There is more than one correct form of each answer.

29. \( 3 \sin^2 x - 5 \sin x - 2 \)
30. \( 6 \cos^2 x + 5 \cos x - 6 \)
31. \( \cot^2 x + \csc x - 1 \)
32. \( \sin^2 x + 3 \cos x + 3 \)

Multiplying Trigonometric Expressions. In Exercises 33 and 34, perform the multiplication and use the fundamental identities to simplify. There is more than one correct form of each answer.

33. \( \sin x + \cos x \)
34. \( 2 \cos x + 2 \)

Simplifying a Trigonometric Expression. In Exercises 35–44, use the fundamental identities to simplify the expression. There is more than one correct form of each answer.

35. \( \cot \theta \sec \theta \)
36. \( \tan(-x) \cos x \)
37. \( \sin \phi \csc \phi - \sin \phi \)
38. \( \cos t(1 + \tan^2 t) \)
39. \( \frac{1 - \sin^2 x}{\csc^2 x - 1} \)
40. \( \tan \theta \cot \theta \)
41. \( \cos \left( \frac{\pi}{2} - x \right) \sec x \)
42. \( \frac{\cos^2 y}{1 - \sin y} \)
43. \( \sin \beta \tan \beta + \cos \beta \)
44. \( \cot u \sin u + \tan u \cos u \)

Adding or Subtracting Trigonometric Expressions. In Exercises 45–48, perform the addition or subtraction and use the fundamental identities to simplify. There is more than one correct form of each answer.

45. \( \frac{1}{1 + \cos x} - \frac{1}{1 - \cos x} \)
46. \( \frac{1}{\sec x + 1} - \frac{1}{\sec x - 1} \)
47. \( \tan x - \frac{\sec^2 x}{\tan x} \)
48. \( \frac{\cos x}{1 + \sin x} - \frac{1 + \sin x}{\cos x} \)

Rewriting a Trigonometric Expression. In Exercises 49 and 50, rewrite the expression so that it is not in fractional form. There is more than one correct form of each answer.

49. \( \frac{\sin^2 y}{1 - \cos y} \)
50. \( \frac{5}{\tan x + \sec x} \)
Trigonometric Functions and Expressions
In Exercises 51 and 52, use a graphing utility to determine which of the six trigonometric functions is equal to the expression. Verify your answer algebraically.

51. \( \cos x \cot x + \sin x \)
52. \( \frac{1}{\sin x \cos x} - \cos x \)

Trigonometric Substitution
In Exercises 53–56, use the trigonometric substitution to write the algebraic expression as a trigonometric function of \( \theta \), where \( 0 < \theta < \pi/2 \).

53. \( \sqrt{9 - x^2} \), \( x = 3 \cos \theta \)
54. \( \sqrt{49 - x^2} \), \( x = 7 \sin \theta \)
55. \( \sqrt{x^2 - 4} \), \( x = 2 \sec \theta \)
56. \( \sqrt{9x^2 + 25} \), \( x = 5 \tan \theta \)

Trigonometric Substitution
In Exercises 57 and 58, use the trigonometric substitution to write the algebraic equation as a trigonometric equation of \( \theta \), where \( -\pi/2 < \theta < \pi/2 \). Then find \( \sin \theta \) and \( \cos \theta \).

57. \( 3 = \sqrt{9 - x^2} \), \( x = 3 \sin \theta \)
58. \( -5 \sqrt{3} = \sqrt{100 - x^2} \), \( x = 10 \cos \theta \)

Solving a Trigonometric Equation
In Exercises 59 and 60, use a graphing utility to solve the equation for \( \theta \), where \( 0 \leq \theta < 2\pi \).

59. \( \sin \theta = \sqrt{1 - \cos^2 \theta} \)
60. \( \sec \theta = \sqrt{1 + \tan^2 \theta} \)

61. Friction
The forces acting on an object weighing \( W \) units on an inclined plane positioned at an angle of \( \theta \) with the horizontal (see figure) are modeled by

\[ \mu W \cos \theta = W \sin \theta \]

where \( \mu \) is the coefficient of friction. Solve the equation for \( \mu \) and simplify the result.

62. Rate of Change
The rate of change of the function \( f(x) = \sec x + \cos x \) is given by the expression \( \sec x \tan x - \sin x \). Show that this expression can also be written as \( \sin x \tan^2 x \).

Exploration
True or False? In Exercises 63 and 64, determine whether the statement is true or false. Justify your answer.

63. The even and odd trigonometric identities are helpful for determining whether the value of a trigonometric function is positive or negative.
64. A cofunction identity can transform a tangent function into a cosecant function.

Finding Limits of Trigonometric Functions
In Exercises 65 and 66, fill in the blanks.

65. As \( x \to \left( \frac{\pi}{2} \right)^- \), \( \tan x \to \) and \( \cot x \to \).
66. As \( x \to \pi \), \( \sin x \to \) and \( \csc x \to \).

Determining Identities
In Exercises 67 and 68, determine whether the equation is an identity, and give a reason for your answer.

67. \( \frac{\sin k\theta}{\cos k\theta} = \tan \theta \) \( k \) is a constant.
68. \( \sin \theta \csc \theta = 1 \)

Trigonometric Substitution
Use the trigonometric substitution \( u = a \tan \theta \), where \( -\pi/2 < \theta < \pi/2 \) and \( a > 0 \), to simplify the expression \( \sqrt{a^2 + u^2} \).

70. HOW DO YOU SEE IT?
Explain how to use the figure to derive the Pythagorean identities

\[ \sin^2 \theta + \cos^2 \theta = 1, \]
\[ 1 + \tan^2 \theta = \sec^2 \theta, \]
and \( 1 + \cot^2 \theta = \csc^2 \theta \).

Discuss how to remember these identities and other fundamental trigonometric identities.

71. Writing Trigonometric Functions in Terms of Sine
Write each of the other trigonometric functions of \( \theta \) in terms of \( \sin \theta \).

72. Rewriting a Trigonometric Expression
Rewrite the following expression in terms of \( \sin \theta \) and \( \cos \theta \).

\[ \frac{\sec \theta (1 - \tan \theta)}{\sec \theta + \csc \theta} \]
2.2 Verifying Trigonometric Identities

■ Verify trigonometric identities.

Introduction

In this section, you will study techniques for verifying trigonometric identities. In the next section, you will study techniques for solving trigonometric equations. The key to verifying identities and solving equations is the ability to use the fundamental identities and the rules of algebra to rewrite trigonometric expressions.

Remember that a conditional equation is an equation that is true for only some of the values in its domain. For example, the conditional equation

\[ \sin x = 0 \]

is true only for

\[ x = n\pi \]

where \( n \) is an integer. When you find these values, you are solving the equation.

On the other hand, an equation that is true for all real values in the domain of the variable is an identity. For example, the familiar equation

\[ \sin^2 x = 1 - \cos^2 x \]

is true for all real numbers \( x \). So, it is an identity.

Verifying Trigonometric Identities

Although there are similarities, verifying that a trigonometric equation is an identity is quite different from solving an equation. There is no well-defined set of rules to follow in verifying trigonometric identities, and it is best to learn the process by practicing.

Guidelines for Verifying Trigonometric Identities

1. Work with one side of the equation at a time. It is often better to work with the more complicated side first.
2. Look for opportunities to factor an expression, add fractions, square a binomial, or create a monomial denominator.
3. Look for opportunities to use the fundamental identities. Note which functions are in the final expression you want. Sines and cosines pair up well, as do secants and tangents, and cosecants and cotangents.
4. If the preceding guidelines do not help, then try converting all terms to sines and cosines.
5. Always try something. Even making an attempt that leads to a dead end can provide insight.

Verifying trigonometric identities is a useful process when you need to convert a trigonometric expression into a form that is more useful algebraically. When you verify an identity, you cannot assume that the two sides of the equation are equal because you are trying to verify that they are equal. As a result, when verifying identities, you cannot use operations such as adding the same quantity to each side of the equation or cross multiplication.
EXAMPLE 1  Verifying a Trigonometric Identity

Verify the identity \( \sec^2 \theta - 1 \overset{\text{sec}^2 \theta}{\text{sec}^2 \theta} = \sin^2 \theta \).

Solution  Start with the left side because it is more complicated.

\[
\frac{\sec^2 \theta - 1}{\sec^2 \theta} = \frac{(\tan^2 \theta + 1) - 1}{\sec^2 \theta} = \frac{\tan^2 \theta}{\sec^2 \theta} = \tan^2 \theta \cos^2 \theta = \sin^2 \theta \cot^2 \theta = \sin^2 \theta
\]

Notice that you verify the identity by starting with the left side of the equation (the more complicated side) and using the fundamental trigonometric identities to simplify it until you obtain the right side.

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Verify the identity \( \frac{\sin^2 \theta + \cos^2 \theta}{\cos^2 \theta \ sec^2 \theta} = 1 \).

There can be more than one way to verify an identity. Here is another way to verify the identity in Example 1.

\[
\frac{\sec^2 \theta - 1}{\sec^2 \theta} = \frac{\sec^2 \theta - 1}{\sec^2 \theta} = 1 - \cos^2 \theta = \sin^2 \theta
\]

EXAMPLE 2  Verifying a Trigonometric Identity

Verify the identity \( 2 \sec^2 \alpha = \frac{1}{1 - \sin \alpha} + \frac{1}{1 + \sin \alpha} \).

Algebraic Solution  Start with the right side because it is more complicated.

\[
\frac{1}{1 - \sin \alpha} + \frac{1}{1 + \sin \alpha} = \frac{1 + \sin \alpha + 1 - \sin \alpha}{(1 - \sin \alpha)(1 + \sin \alpha)} = \frac{2}{1 - \sin^2 \alpha} = \frac{2}{\cos^2 \alpha} = 2 \sec^2 \alpha
\]

Numerical Solution  Use a graphing utility to create a table that shows the values of \( y_1 = 2/\cos^2 x \) and \( y_2 = 1/(1 - \sin x) + 1/(1 + \sin x) \) for different values of \( x \).

<table>
<thead>
<tr>
<th>( x )</th>
<th>( y_1 )</th>
<th>( y_2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>(-\pi/2)</td>
<td>2.5966</td>
<td>2.5966</td>
</tr>
<tr>
<td>0</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>( \pi/4 )</td>
<td>2.1304</td>
<td>2.1304</td>
</tr>
<tr>
<td>( \pi/2 )</td>
<td>2.1304</td>
<td>2.1304</td>
</tr>
<tr>
<td>( \pi )</td>
<td>2.5966</td>
<td>2.5966</td>
</tr>
<tr>
<td>( 3\pi/2 )</td>
<td>2.7557</td>
<td>2.7557</td>
</tr>
<tr>
<td>( 2\pi )</td>
<td>2.8531</td>
<td>2.8531</td>
</tr>
</tbody>
</table>

The values for \( y_1 \) and \( y_2 \) appear to be identical, so the equation appears to be an identity.

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Verify the identity \( 2 \csc^2 \beta = \frac{1}{1 - \cos \beta} + \frac{1}{1 + \cos \beta} \).
In Example 2, you needed to write the Pythagorean identity \( \sin^2 \theta + \cos^2 \theta = 1 \) in the equivalent form \( \cos^2 \theta = 1 - \sin^2 \theta \). When verifying identities, you may find it useful to write the Pythagorean identities in one of these equivalent forms.

### Pythagorean Identities

- \( \sin^2 \theta + \cos^2 \theta = 1 \)
- \( \cos^2 \theta = 1 - \sin^2 \theta \)
- \( 1 = \sec^2 \theta - \tan^2 \theta \)
- \( \tan^2 \theta = \sec^2 \theta - 1 \)
- \( 1 + \cot^2 \theta = \csc^2 \theta \)
- \( \cot^2 \theta = \csc^2 \theta - 1 \)

### Example 3

**Verifying a Trigonometric Identity**

Verify the identity \( (\tan^2 x + 1)(\cos^2 x - 1) = -\tan^2 x \).

#### Algebraic Solution

By applying identities before multiplying, you obtain the following.

\[
(tan^2 x + 1)(cos^2 x - 1) = (sec^2 x - \sin^2 x)(\cos^2 x - 1)
\]

\[
= \frac{\sin^2 x}{\cos^2 x} 
\]

\[
= \left( \frac{\sin x}{\cos x} \right)^2 
\]

\[
= -\tan^2 x 
\]

### Graphical Solution

Because the graphs appear to coincide, the given equation appears to be an identity.

#### Checkpoint

Verify the identity \( (\sec^2 x - 1)(\sin^2 x - 1) = -\sin^2 x \).

### Remark

Although a graphing utility can be useful in helping to verify an identity, you must use algebraic techniques to produce a valid proof.

### Example 4

**Converting to Sines and Cosines**

Verify the identity \( \tan x + \cot x = \sec x \csc x \).

#### Solution

Convert the left side into sines and cosines.

\[
\tan x + \cot x = \frac{\sin x}{\cos x} + \frac{\cos x}{\sin x} 
\]

\[
= \frac{\sin^2 x + \cos^2 x}{\cos x \sin x} 
\]

\[
= \frac{1}{\cos x \sin x} 
\]

\[
= \frac{1}{\cos x} \cdot \frac{1}{\sin x} 
\]

\[
= \sec x \csc x 
\]

#### Checkpoint

Verify the identity \( \csc x - \sin x = \cos x \cot x \).
Recall from algebra that rationalizing the denominator using conjugates is, on occasion, a powerful simplification technique. A related form of this technique works for simplifying trigonometric expressions as well. For instance, to simplify

\[
\frac{1}{1 - \cos \alpha}
\]

multiply the numerator and the denominator by \(1 + \cos \alpha\).

\[
\frac{1}{1 - \cos \alpha} = \frac{1}{1 - \cos \alpha} \left( \frac{1 + \cos \alpha}{1 + \cos \alpha} \right)
\]

\[
= \frac{1 + \cos \alpha}{1 - \cos^2 \alpha}
\]

\[
= \frac{1 + \cos \alpha}{\sin^2 \alpha}
\]

\[
= \csc^2 \alpha (1 + \cos \alpha)
\]

The expression \(\csc^2 \alpha (1 + \cos \alpha)\) is considered a simplified form of

\[
\frac{1}{1 - \cos \alpha}
\]

because \(\csc^2 \alpha (1 + \cos \alpha)\) does not contain fractions.

**EXAMPLE 5**

**Verifying a Trigonometric Identity**

Verify the identity \(\sec \alpha + \tan \alpha = \frac{\cos \alpha}{1 - \sin \alpha}\).

**Algebraic Solution**

Begin with the right side and create a monomial denominator by multiplying the numerator and the denominator by \(1 + \sin \alpha\).

\[
\frac{\cos \alpha}{1 - \sin \alpha} = \frac{\cos \alpha}{1 - \sin \alpha} \left( \frac{1 + \sin \alpha}{1 + \sin \alpha} \right)
\]

\[
= \frac{\cos \alpha + \cos \alpha \sin \alpha}{1 - \sin^2 \alpha}
\]

\[
= \frac{\cos \alpha + \cos \alpha \sin \alpha}{\cos^2 \alpha}
\]

\[
= \frac{\cos \alpha}{\cos \alpha} + \frac{\sin \alpha}{\cos \alpha}
\]

\[
= \sec \alpha + \tan \alpha
\]

**Graphical Solution**

Because the graphs appear to coincide, the given equation appears to be an identity.

**Checkpoint**

Verify the identity \(\csc \alpha + \cot \alpha = \frac{\sin \alpha}{1 - \cos \alpha}\).

In Examples 1 through 5, you have been verifying trigonometric identities by working with one side of the equation and converting to the form given on the other side. On occasion, it is practical to work with each side separately, to obtain one common form that is equivalent to both sides. This is illustrated in Example 6.
EXAMPLE 6  Working with Each Side Separately

Verify the identity \( \frac{\cot^2 \theta}{1 + \csc \theta} = \frac{1 - \sin \theta}{\sin \theta} \).

**Algebraic Solution**

Working with the left side, you have

\[
\frac{\cot^2 \theta}{1 + \csc \theta} = \frac{\csc^2 \theta - 1}{1 + \csc \theta} = \frac{(\csc \theta - 1)(\csc \theta + 1)}{1 + \csc \theta} = \csc \theta - 1.
\]

Now, simplifying the right side, you have

\[
\frac{1 - \sin \theta}{\sin \theta} = \frac{1}{\sin \theta} - \frac{\sin \theta}{\sin \theta} = \csc \theta - 1.
\]

This verifies the identity because both sides are equal to \( \csc \theta - 1 \).

✓ **Checkpoint** (Audio-video solution in English & Spanish at LarsonPrecalculus.com)

Verify the identity \( \frac{\tan^2 \theta}{1 + \sec \theta} = \frac{1 - \cos \theta}{\cos \theta} \).

**Numerical Solution**

Use a graphing utility to create a table that shows the values of

\[
y_1 = \frac{\cot^2 x}{1 + \csc x} \quad \text{and} \quad y_2 = \frac{1 - \sin x}{\sin x}
\]

for different values of \( x \).

<table>
<thead>
<tr>
<th>x</th>
<th>y_1</th>
<th>y_2</th>
</tr>
</thead>
<tbody>
<tr>
<td>-1.5</td>
<td>-2.084</td>
<td>2.084</td>
</tr>
<tr>
<td>-1.25</td>
<td>-1.942</td>
<td>1.942</td>
</tr>
<tr>
<td>-1.0</td>
<td>-1.001</td>
<td>1.001</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0.25</td>
<td>3.042</td>
<td>-3.042</td>
</tr>
<tr>
<td>-1</td>
<td>1.004</td>
<td>-1.004</td>
</tr>
<tr>
<td>0.25</td>
<td>-2.084</td>
<td>2.084</td>
</tr>
</tbody>
</table>

The values for \( y_1 \) and \( y_2 \) appear to be identical, so the equation appears to be an identity.

EXAMPLE 7  Two Examples from Calculus

Verify each identity.

a. \( \tan^4 x = \tan^2 x \sec^2 x - \tan^2 x \)

b. \( \csc^4 x \cot x = \csc^2 x \cot x + \cot^3 x \)

**Solution**

a. \( \tan^4 x = (\tan^2 x)(\tan^2 x) \)

\[
= \tan^2 x(\sec^2 x - 1) \quad \text{Pythagorean identity}
= \tan^2 x \sec^2 x - \tan^2 x \quad \text{Multiply.}
\]

b. \( \csc^4 x \cot x = \csc^2 x \csc^2 x \cot x \)

\[
= \csc^2 x(1 + \cot^2 x) \cot x \quad \text{Pythagorean identity}
= \csc^2 x \cot x + \cot^3 x \quad \text{Multiply.}
\]

✓ **Checkpoint** (Audio-video solution in English & Spanish at LarsonPrecalculus.com)

Verify each identity.

a. \( \tan^3 x = \tan x \sec^2 x - \tan x \)

b. \( \sin^3 x \cos^2 x = \cos^4 x - \cos^6 x \sin x \)

**Summarize** (Section 2.2)

1. State the guidelines for verifying trigonometric identities (page 217). For examples of verifying trigonometric identities, see Examples 1–7.
2.2 Exercises

Vocabulary

In Exercises 1 and 2, fill in the blanks.

1. An equation that is true for all real values in its domain is called an __________.
2. An equation that is true for only some values in its domain is called a __________.

In Exercises 3–8, fill in the blank to complete the fundamental trigonometric identity.

3. \( \frac{1}{\cot u} = \) __________
4. \( \frac{\cos u}{\sin u} = \) __________
5. \( \sin^2 u + \) __________ = 1
6. \( \cos \left( \frac{\pi}{2} - u \right) = \) __________
7. \( \csc(-u) = \) __________
8. \( \sec(-u) = \) __________

Skills and Applications

Verifying a Trigonometric Identity  In Exercises 9–50, verify the identity.

9. \( \tan t \cot t = 1 \)
10. \( \sec y \cos y = 1 \)
11. \( \cot^2 y (\sec^2 y - 1) = 1 \)
12. \( \cos x + \sin x \tan x = \sec x \)
13. \( (1 + \sin \alpha)(1 - \sin \alpha) = \cos^2 \alpha \)
14. \( \cos^2 \beta - \sin^2 \beta = 2 \cos^2 \beta - 1 \)
15. \( \cos^2 \beta - \sin^2 \beta = 1 - 2 \sin^2 \beta \)
16. \( \sin^2 \alpha - \sin^4 \alpha = \cos^2 \alpha - \cos^4 \alpha \)
17. \( \tan^2 \theta = \sec \theta \tan \theta \)
18. \( \frac{\cot^2 t}{\csc t} = \cos t(\sec^2 t - 1) \)
19. \( \frac{\cot^2 t}{\csc t} = \frac{1 - \sin^2 t}{\sin t} \)
20. \( \frac{1}{\tan \beta} - \tan \beta = \frac{\sec^2 \beta}{\tan \beta} \)
21. \( \sin^{-1/2} x \cos x - \sin^{-3/2} x \cos x = \cos^3 x \sqrt{\sin x} \)
22. \( \sec^2 x \tan x - \sec^4 x \sin x \tan x = \sec^3 x \tan^2 x \)
23. \( \cot x \sec x = \csc x - \sin x \)
24. \( \frac{\sec \theta - 1}{1 - \cos \theta} = \sec \theta \)
25. \( \sec x - \cos x = \sin x \tan x \)
26. \( \sec x \csc x - 2 \sin x = \cot x - \tan x \)
27. \( \frac{1}{\tan x} + \frac{1}{\cot x} = \tan x + \cot x \)
28. \( \frac{1}{\sin x} - \frac{1}{\csc x} = \csc x - \sin x \)
29. \( \frac{1 - \sin \theta}{\cos \theta} + \frac{\cos \theta}{1 + \sin \theta} = 2 \sec \theta \)
30. \( \frac{\cos \theta \cot \theta}{1 - \sin \theta} - 1 = \csc \theta \)
31. \( \frac{1}{\cos x} + 1 + \frac{1}{\cos x - 1} = -2 \csc x \cot x \)
32. \( \cos x - \frac{\cos x}{1 + \tan x} = \frac{\sin x \cos x}{\sin x - \cos x} \)
33. \( \tan \left( \frac{\pi}{2} - \theta \right) \tan \theta = 1 \)
34. \( \frac{\cos(\pi/2 - x)}{\sin(\pi/2 - x)} = \tan x \)
35. \( \tan x \cot x = \sec x \)
36. \( \frac{\csc(-x)}{\sec(-x)} = -\cot x \)
37. \( (1 + \sin y)(1 + \sin(-y)) = \cos^2 y \)
38. \( \tan x \cot y = \cot x \cot y \)
39. \( \tan x \cot y = \tan y + \cot x \)
40. \( \cos x - \cos y = \frac{\sin x - \sin y}{\sin x - \cos y} = 0 \)
41. \( \sqrt{\frac{1 + \sin \theta}{1 - \sin \theta}} = \frac{1 + \sin \theta}{\cos \theta} \)
42. \( \sqrt{\frac{1 - \cos \theta}{1 + \cos \theta}} = \frac{1 - \cos \theta}{\sin \theta} \)
43. \( \cos^2 \beta + \cos^2 \left( \frac{\pi}{2} - \beta \right) = 1 \)
44. \( \sec^2 y - \cot^2 \left( \frac{\pi}{2} - y \right) = 1 \)
45. \( \sin \theta \csc \left( \frac{\pi}{2} - \theta \right) = \tan \theta \)
46. \( \sec^2 \left( \frac{\pi}{2} - x \right) - 1 = \cot^2 x \)
47. \( \tan(\sin^{-1} x) = \frac{x}{\sqrt{1 - x^2}} \)
48. \( \cos(\sin^{-1} x) = \sqrt{1 - x^2} \)
49. \( \tan \left( \sin^{-1} \frac{x - 1}{x} \right) = \frac{x - 1}{\sqrt{16 - (x - 1)^2}} \)
50. \( \tan(\cos^{-1} \frac{x - 1}{2}) = \frac{\sqrt{4 - (x - 1)^2}}{x - 1} \)
2.2 Verifying Trigonometric Identities

66. Shadow Length

The length $s$ of a shadow cast by a vertical gnomon (a device used to tell time) of height $h$ when the angle of the sun above the horizon is $\theta$ can be modeled by the equation

$$s = \frac{h \sin(90^\circ - \theta)}{\sin \theta}.$$

(a) Verify that the expression for $s$ is equal to $h \cot \theta$.

(b) Use a graphing utility to complete the table. Let $h = 5$ feet.

<table>
<thead>
<tr>
<th>$\theta$</th>
<th>$15^\circ$</th>
<th>$30^\circ$</th>
<th>$45^\circ$</th>
<th>$60^\circ$</th>
<th>$75^\circ$</th>
<th>$90^\circ$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$s$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

(c) Use your table from part (b) to determine the angles of the sun that result in the maximum and minimum lengths of the shadow.

(d) Based on your results from part (c), what time of day do you think it is when the angle of the sun above the horizon is $90^\circ$?

Exploration

True or False? In Exercises 67–69, determine whether the statement is true or false. Justify your answer.

67. There can be more than one way to verify a trigonometric identity.

68. The equation $\sin^2 \theta + \cos^2 \theta = 1 + \tan^2 \theta$ is an identity because $\sin^2(0) + \cos^2(0) = 1$ and $1 + \tan^2(0) = 1$.

69. $\sin x^2 = \sin x$

70. HOW DO YOU SEE IT? Explain how to use the figure to derive the identity

$$\frac{\sec^2 \theta - 1}{\sec^2 \theta} = \sin^2 \theta$$

given in Example 1.

Think About It In Exercises 71–74, explain why the equation is not an identity and find one value of the variable for which the equation is not true.

71. $\sin \theta = \sqrt{1 - \cos^2 \theta}$

72. $\tan \theta = \sqrt{\sec^2 \theta - 1}$

73. $\frac{1}{1 + \cos \theta} = \sin \theta$

74. $1 - \tan \theta = \sec \theta$
2.3 Solving Trigonometric Equations

- Use standard algebraic techniques to solve trigonometric equations.
- Solve trigonometric equations of quadratic type.
- Solve trigonometric equations involving multiple angles.
- Use inverse trigonometric functions to solve trigonometric equations.

Introduction

To solve a trigonometric equation, use standard algebraic techniques (when possible) such as collecting like terms and factoring. Your preliminary goal in solving a trigonometric equation is to isolate the trigonometric function on one side of the equation. For example, to solve the equation \(2 \sin x = 1\), divide each side by 2 to obtain

\[
\sin x = \frac{1}{2}.
\]

To solve for \(x\), note in the figure below that the equation \(\sin x = \frac{1}{2}\) has solutions \(x = \frac{\pi}{6}\) and \(x = \frac{5\pi}{6}\) in the interval \([0, 2\pi]\). Moreover, because \(\sin x\) has a period of \(2\pi\), there are infinitely many other solutions, which can be written as

\[
x = \frac{\pi}{6} + 2n\pi \quad \text{and} \quad x = \frac{5\pi}{6} + 2n\pi
\]

where \(n\) is an integer, as shown below.

![Graph of sine function with solutions for \(x = \frac{\pi}{6}, x = \frac{5\pi}{6}\) and \(x = \frac{7\pi}{6}\)]

The figure below illustrates another way to show that the equation \(\sin x = \frac{1}{2}\) has infinitely many solutions. Any angles that are coterminal with \(\frac{\pi}{6}\) or \(\frac{5\pi}{6}\) will also be solutions of the equation.

\[
\sin \left(\frac{5\pi}{6} + 2n\pi\right) = \frac{1}{2}
\]

When solving trigonometric equations, you should write your answer(s) using exact values, when possible, rather than decimal approximations.
2.3 Solving Trigonometric Equations

### EXAMPLE 1 Collecting Like Terms

Solve

\[ \sin x + \sqrt{2} = -\sin x. \]

**Solution** Begin by isolating \( \sin x \) on one side of the equation.

\[
\begin{align*}
\sin x + \sqrt{2} &= -\sin x \\
\sin x + \sin x + \sqrt{2} &= 0 \\
\sin x + \sin x &= -\sqrt{2} \\
2 \sin x &= -\sqrt{2} \\
\sin x &= -\frac{\sqrt{2}}{2}
\end{align*}
\]

Write original equation.

Add \( \sin x \) to each side.

Subtract \( \sqrt{2} \) from each side.

Combine like terms.

Divide each side by 2.

Because \( \sin x \) has a period of \( 2\pi \), first find all solutions in the interval \([0, 2\pi]\). These solutions are \( x = \frac{5\pi}{4} \) and \( x = \frac{7\pi}{4} \). Finally, add multiples of \( 2\pi \) to each of these solutions to obtain the general form

\[ x = \frac{5\pi}{4} + 2n\pi \quad \text{and} \quad x = \frac{7\pi}{4} + 2n\pi \]

where \( n \) is an integer.

**Checkpoint** [Audio-video solution in English & Spanish at LarsonPrecalculus.com]

Solve \( \sin x - \sqrt{2} = -\sin x \).

### EXAMPLE 2 Extracting Square Roots

Solve

\[ 3 \tan^2 x - 1 = 0. \]

**Solution** Begin by isolating \( \tan x \) on one side of the equation.

\[
\begin{align*}
3 \tan^2 x - 1 &= 0 \\
3 \tan^2 x &= 1 \\
\tan^2 x &= \frac{1}{3} \\
\tan x &= \pm \frac{1}{\sqrt{3}} \\
\tan x &= \frac{\pm \sqrt{3}}{3}
\end{align*}
\]

Write original equation.

Add 1 to each side.

Divide each side by 3.

Extract square roots.

Rationalize the denominator.

Because \( \tan x \) has a period of \( \pi \), first find all solutions in the interval \([0, \pi]\). These solutions are \( x = \frac{\pi}{6} \) and \( x = \frac{5\pi}{6} \). Finally, add multiples of \( \pi \) to each of these solutions to obtain the general form

\[ x = \frac{\pi}{6} + n\pi \quad \text{and} \quad x = \frac{5\pi}{6} + n\pi \]

where \( n \) is an integer.

**Checkpoint** [Audio-video solution in English & Spanish at LarsonPrecalculus.com]

Solve \( 4 \sin^2 x - 3 = 0. \)
The equations in Examples 1 and 2 involved only one trigonometric function. When two or more functions occur in the same equation, collect all terms on one side and try to separate the functions by factoring or by using appropriate identities. This may produce factors that yield no solutions, as illustrated in Example 3.

**Example 3**

*Factoring*

Solve \( \cot x \cos^2 x = 2 \cot x \).

**Solution**

Begin by collecting all terms on one side of the equation and factoring:

\[
\cot x \cos^2 x = 2 \cot x \\
\cot x \cos^2 x - 2 \cot x = 0 \\
\cot x(\cos^2 x - 2) = 0
\]

Write original equation.

Subtract 2 cot x from each side.

Factor.

By setting each of these factors equal to zero, you obtain:

\[
\cot x = 0 \quad \text{and} \quad \cos^2 x - 2 = 0
\]

\[
\cos^2 x = 2
\]

\[
\cos x = \pm \sqrt{2}.
\]

In the interval \((0, \pi)\), the equation \( \cot x = 0 \) has the solution

\[
x = \frac{\pi}{2}.
\]

No solution exists for \( \cos x = \pm \sqrt{2} \) because \( \pm \sqrt{2} \) are outside the range of the cosine function. Because \( \cot x \) has a period of \( \pi \), you obtain the general form of the solution by adding multiples of \( \pi \) to \( x = \pi/2 \) to get

\[
x = \frac{\pi}{2} + \pi n
\]

General solution

where \( n \) is an integer. Confirm this graphically by sketching the graph of \( y = \cot x \cos^2 x - 2 \cot x \), as shown below.

Notice that the \( x \)-intercepts occur at

\[
-\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \quad \text{and so on. These \( x \)-intercepts correspond to the solutions of} \quad \cot x \cos^2 x - 2 \cot x = 0.
\]

**Checkpoint**

Solve \( \sin^2 x = 2 \sin x \).
2.3  Solving Trigonometric Equations

Equations of Quadratic Type

Many trigonometric equations are of quadratic type $ax^2 + bx + c = 0$, as shown below. To solve equations of this type, factor the quadratic or, when this is not possible, use the Quadratic Formula.

<table>
<thead>
<tr>
<th>Quadratic in $\sin x$</th>
<th>Quadratic in $\sec x$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$2 \sin^2 x - \sin x - 1 = 0$</td>
<td>$\sec^2 x - 3 \sec x - 2 = 0$</td>
</tr>
<tr>
<td>$(2 \sin x + 1)(\sin x - 1) = 0$</td>
<td>$(\sec x)^2 - 3(\sec x) - 2 = 0$</td>
</tr>
</tbody>
</table>

**Example 1**  
Factoring an Equation of Quadratic Type

Find all solutions of $2 \sin^2 x - \sin x - 1 = 0$ in the interval $[0, 2\pi]$.

**Algebraic Solution**

Treat the equation as a quadratic in $\sin x$ and factor.

$$2 \sin^2 x - \sin x - 1 = 0$$

Write original equation.

$$(2 \sin x + 1)(\sin x - 1) = 0$$

Factor.

Setting each factor equal to zero, you obtain the following solutions in the interval $[0, 2\pi]$.

- $2 \sin x + 1 = 0$ and $\sin x - 1 = 0$
- $\sin x = -\frac{1}{2}$
- $\sin x = 1$

$$x = \frac{7\pi}{6}, \frac{11\pi}{6}, \frac{\pi}{2}$$

**Graphical Solution**

From the above figure, you can conclude that the approximate solutions of $2 \sin^2 x - \sin x - 1 = 0$ in the interval $[0, 2\pi]$ are $x = 1.571, x = 3.665, x = 5.760, x = 1.571 + \frac{\pi}{6}, x = 3.665 - \frac{\pi}{6}, x = 5.760 - \frac{\pi}{6}$.

**Checkpoint**

Find all solutions of $2 \sin^2 x - 3 \sin x + 1 = 0$ in the interval $[0, 2\pi]$.

**Example 2**  
Rewriting with a Single Trigonometric Function

Solve $2 \sin^2 x + 3 \cos x - 3 = 0$.

**Solution**

This equation contains both sine and cosine functions. You can rewrite the equation so that it has only cosine functions by using the identity $\sin^2 x = 1 - \cos^2 x$.

$$2 \sin^2 x + 3 \cos x - 3 = 0$$

Write original equation.

$$2(1 - \cos^2 x) + 3 \cos x - 3 = 0$$

Pythagorean identity.

$$2 \cos^2 x - 3 \cos x + 1 = 0$$

Multiply each side by $-1$.

$$2 \cos x - 1)(\cos x - 1) = 0$$

Factor.

By setting each factor equal to zero, you can find the solutions in the interval $[0, 2\pi)$ to be $x = 0, x = \pi/3$, and $x = 5\pi/3$. Because $\cos x$ has a period of $2\pi$, the general solution is

$$x = 2n\pi, \quad x = \frac{\pi}{3} - 2n\pi, \quad x = \frac{5\pi}{3} + 2n\pi$$

where $n$ is an integer.

**Checkpoint**

Solve $3 \sec^2 x - 2 \tan^2 x - 4 = 0$.

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Sometimes you must square each side of an equation to obtain a quadratic, as demonstrated in the next example. Because this procedure can introduce extraneous solutions, you should check any solutions in the original equation to see whether they are valid or extraneous.

**EXAMPLE 6**  
**Squaring and Converting to Quadratic Type**

Find all solutions of \( \cos x + 1 = \sin x \) in the interval \([0, 2\pi]\).

**Solution**  
It is not clear how to rewrite this equation in terms of a single trigonometric function. Notice what happens when you square each side of the equation.

\[
\begin{align*}
\cos x + 1 &= \sin x \\
\cos^2 x + 2 \cos x - 1 &= \sin^2 x \\
\cos^2 x + 2 \cos x + 1 &= 1 - \cos^2 x \\
\cos^2 x + \cos^2 x + 2 \cos x + 1 - 1 &= 0 \\
2 \cos^2 x + 2 \cos x &= 0 \\
2 \cos x(\cos x + 1) &= 0
\end{align*}
\]

Setting each factor equal to zero produces

\[
2 \cos x = 0 \quad \text{and} \quad \cos x + 1 = 0
\]

\[
\begin{align*}
\cos x &= 0 \\
\cos x &= -1
\end{align*}
\]

\[
\begin{align*}
x &= \frac{\pi}{2}, \frac{3\pi}{2} \\
x &= \pi
\end{align*}
\]

Because you squared the original equation, check for extraneous solutions.

**Check \( x = \frac{\pi}{2} \)**

\[
\begin{align*}
\cos \frac{\pi}{2} + 1 &= \sin \frac{\pi}{2} \\
0 + 1 &= 1
\end{align*}
\]

Solution checks. \( \checkmark \)

**Check \( x = \frac{3\pi}{2} \)**

\[
\begin{align*}
\cos \frac{3\pi}{2} + 1 &= \sin \frac{3\pi}{2} \\
0 + 1 &= -1
\end{align*}
\]

Solution does not check.

**Check \( x = \pi \)**

\[
\begin{align*}
\cos \pi + 1 &= \sin \pi \\
-1 + 1 &= 0
\end{align*}
\]

Solution checks. \( \checkmark \)

Of the three possible solutions, \( x = 3\pi/2 \) is extraneous. So, in the interval \([0, 2\pi]\), the only two solutions are

\[
\begin{align*}
x &= \frac{\pi}{2} \\
x &= \pi
\end{align*}
\]

\( \checkmark \) **Checkpoint**  
Find all solutions of \( \sin x + 1 = \cos x \) in the interval \([0, 2\pi]\).
Functions Involving Multiple Angles

The next two examples involve trigonometric functions of multiple angles of the forms \( \cos k\theta \) and \( \tan k\theta \). To solve equations of these forms, first solve the equation for \( k\theta \) and then divide your result by \( k \).

**EXAMPLE 7** Solving a Multiple-Angle Equation

Solve \( 2 \cos 3\theta - 1 = 0 \).

**Solution**

\[
2 \cos 3\theta - 1 = 0 \\
2 \cos 3\theta = 1 \\
\cos 3\theta = \frac{1}{2}
\]

Write original equation.

Add 1 to each side.

Divide each side by 2.

In the interval \([0, 2\pi]\), you know that \( 3\theta = \pi/3 \) and \( 3\theta = 5\pi/3 \) are the only solutions, so, in general, you have

\( 3\theta = \frac{\pi}{3} + 2n\pi \) and \( 3\theta = \frac{5\pi}{3} + 2n\pi \).

Dividing these results by 3, you obtain the general solution

\( \theta = \frac{\pi}{9} + \frac{2n\pi}{3} \) and \( \theta = \frac{5\pi}{9} + \frac{2n\pi}{3} \)

General solution

where \( n \) is an integer.

**Checkpoint**

Solve \( 2 \sin 2\theta - \sqrt{3} = 0 \).

**EXAMPLE 8** Solving a Multiple-Angle Equation

\( 3 \tan \frac{x}{2} + 3 = 0 \)

**Original equation**

\[
3 \tan \frac{x}{2} = -3 \\
\tan \frac{x}{2} = -1
\]

Subtract 3 from each side.

Divide each side by 3.

In the interval \([0, \pi]\), you know that \( x/2 = 3\pi/4 \) is the only solution, so, in general, you have

\( x = \frac{3\pi}{4} + n\pi \).

Multiplying this result by 2, you obtain the general solution

\( x = \frac{3\pi}{2} + 2n\pi \)

General solution

where \( n \) is an integer.

**Checkpoint**

Solve \( 2 \tan \frac{x}{2} + 2 = 0 \).
Using Inverse Functions

**Example 9** Using Inverse Functions

\[ \sec^2 x - 2 \tan x = 4 \]

Original equation

\[ 1 + \tan^2 x - 2 \tan x - 4 = 0 \]

Pythagorean identity

\[ \tan^2 x - 2 \tan x - 3 = 0 \]

Combine like terms.

\[ (\tan x - 3)(\tan x + 1) = 0 \]

Factor

Setting each factor equal to zero, you obtain two solutions in the interval \((-\pi/2, \pi/2)\).

\[ x = \arctan 3 \quad \text{and} \quad x = \arctan(-1) = -\pi/4 \]

[Recall that the range of the inverse tangent function is \((-\pi/2, \pi/2)\).]

Finally, because \(\tan x\) has a period of \(\pi\), you add multiples of \(\pi\) to obtain

\[ x = \arctan 3 + n\pi \quad \text{and} \quad x = (-\pi/4 + n\pi) \]  
General solution

where \(n\) is an integer. You can use a calculator to approximate the value of \(\arctan 3\).

**Checkpoint** Audio-video solution in English & Spanish at LarsonPrecalculus.com.

Solve \(4 \tan^2 x + 5 \tan x - 6 = 0\).

Surface Area of a Honeycomb Cell

The surface area \(S\) (in square inches) of a honeycomb cell is given by

\[ S = 6hs + 4.5s^2 \left[ \frac{\sqrt{3} - \cos \theta}{\sin \theta} \right] \]

\[ 0 < \theta \leq 90^\circ \]

where \(h = 2.4\) inches, \(s = 0.75\) inch, and \(\theta\) is the angle shown in Figure 2.1. What value of \(\theta\) gives the minimum surface area?

**Solution** Letting \(h = 2.4\) and \(s = 0.75\), you obtain

\[ S = 10.8 + 0.84375 \left[ \frac{\sqrt{3} - \cos \theta}{\sin \theta} \right] \]

Graph this function using a graphing utility. The minimum point on the graph, which occurs at \(\theta \approx 54.7^\circ\), is shown in Figure 2.2. By using calculus, the exact minimum point on the graph can be shown to occur at \(\theta = \arccos(1/\sqrt{3}) = 0.9553 \approx 54.7356^\circ\).

**Checkpoint** Audio-video solution in English & Spanish at LarsonPrecalculus.com.

In Example 10, for what value(s) of \(\theta\) is the surface area 12 square inches?

Summarize (Section 2.3)

1. Describe how to use standard algebraic techniques to solve trigonometric equations (page 224). For examples of using standard algebraic techniques to solve trigonometric equations, see Examples 1–3.

2. Explain how to solve a trigonometric equation of quadratic type (page 227). For examples of solving trigonometric equations of quadratic type, see Examples 4–6.

3. Explain how to solve a trigonometric equation involving multiple angles (page 229). For examples of solving trigonometric equations involving multiple angles, see Examples 7 and 8.

4. Explain how to use inverse trigonometric functions to solve trigonometric equations (page 230). For examples of using inverse trigonometric functions to solve trigonometric equations, see Examples 9 and 10.
2.3 Exercises

Vocabulary: Fill in the blanks.

1. When solving a trigonometric equation, the preliminary goal is to ________ the trigonometric function involved in the equation.

2. The equation \(2 \sin \theta - 1 = 0\) has the solutions \(\theta = \frac{7\pi}{6} + 2n\pi\) and \(\theta = \frac{11\pi}{6} + 2n\pi\), which are called ________ solutions.

3. The equation \(2 \tan^2 x - 3 \tan x + 1 = 0\) is a trigonometric equation that is of ________ type.

4. A solution of an equation that does not satisfy the original equation is called an ________ solution.

Skills and Applications

Verifying Solutions In Exercises 5–10, verify that the \(x\)-values are solutions of the equation.

5. \(\tan x - \sqrt{3} = 0\)
   (a) \(x = \frac{\pi}{3}\)
   (b) \(x = \frac{4\pi}{3}\)
6. \(\sec x - 2 = 0\)
   (a) \(x = \frac{\pi}{3}\)
   (b) \(x = \frac{5\pi}{3}\)

7. \(3 \tan^2 x - 1 = 0\)
   (a) \(x = \frac{\sqrt{2}}{12}\)
   (b) \(x = \frac{5\sqrt{2}}{12}\)
8. \(2 \cos^2 4x - 1 = 0\)
   (a) \(x = \frac{7\pi}{6}\)
   (b) \(x = \frac{\pi}{6}\)

9. \(2 \sin^2 x - \sin x - 1 = 0\)
   (a) \(x = \frac{\pi}{2}\)
   (b) \(x = \frac{7\pi}{6}\)

10. \(\csc x - 4 \csc^2 x = 0\)
    (a) \(x = \frac{\pi}{6}\)
    (b) \(x = \frac{5\pi}{6}\)

Solving a Trigonometric Equation In Exercises 11–24, solve the equation.

11. \(\sqrt{3} \csc x - 2 = 0\)

12. \(\tan x + \sqrt{3} = 0\)

13. \(\cos x + 1 = -\cos x\)

14. \(3 \sin x + 1 = \sin x\)

15. \(3 \sec^2 x - 4 = 0\)

16. \(3 \cot^2 x - 1 = 0\)

17. \(4 \cos^2 x - 1 = 0\)

18. \(\sin^2 x = 3 \cos^2 x\)

19. \(2 \sin^2 2x = 1\)

20. \(\tan^2 3x = 3\)

21. \(\tan 3x + \tan x - 1 = 0\)

22. \(\cos 2x + \cos x + 1 = 0\)

23. \(\sin x \sin x - 1 = 0\)

24. \((2 \sin^2 x - 1)(\tan^2 x - 3) = 0\)

Solving a Trigonometric Equation In Exercises 25–38, find all solutions of the equation in the interval \([0, 2\pi]\).

25. \(\cos^3 x = \cos x\)

26. \(\sec^3 x - 1 = 0\)

27. \(3 \tan^3 x = \tan x\)

28. \(2 \sin^3 x = 2 + \cos x\)

29. \(\sec^2 x - \sec x = 2\)

30. \(\sec x \csc x = 2 \csc x\)

31. \(2 \sin x + \csc x = 0\)

32. \(\sin x - 2 = \cos x - 2\)

33. \(\cos^2 x + \cos x = 0\)

34. \(2 \sin^2 x + 3 \sin x + 1 = 0\)

35. \(2 \cos^2 x + \tan^2 x = 3\)

36. \(\cos x + \sin x \tan x = 2\)

37. \(\csc x + \cot x = 1\)

38. \(\sec x + \tan x = 1\)

Solving a Multiple-Angle Equation In Exercises 39–44, solve the multiple-angle equation.

39. \(2 \cos 2x + \frac{\sqrt{3}}{2} = 0\)

40. \(2 \sin 2x + \sqrt{3} = 0\)

41. \(\tan 3x = 1\)

42. \(\sec 4x - 2 = 0\)

43. \(2 \cos \frac{x}{2} - \sqrt{2} = 0\)

44. \(2 \sin \frac{x}{2} + \sqrt{3} = 0\)

Finding x-intercepts In Exercises 45–48, find the x-intercepts of the graph.

45. \(y = \sin \frac{\pi x}{2} + 1\)

46. \(y = \sin \pi x + \cos \pi x\)

47. \(y = \tan \left(\frac{\pi x}{6}\right) - 3\)

48. \(y = \sec \left(\frac{\pi x}{8}\right) - 4\)
Approximating Solutions: In Exercises 49–58, use a graphing utility to approximate the solutions (to three decimal places) of the equation in the interval [0, 2π).

49. \(2 \sin x + \cos x = 0\)
50. \(4 \sin^2 x + 2 \sin x - 2 \sin x - 1 = 0\)
51. \(\frac{1 + \sin x}{\cos x} + \frac{\cos x}{1 + \sin x} = 4\)
52. \(\frac{\cos x \cot x}{1 - \sin x} = 3\)
53. \(x \tan x - 1 = 0\)
54. \(x \cos x - 1 = 0\)
55. \(\sec^2 x + 0.5 \tan x - 1 = 0\)
56. \(\csc^2 x + 0.5 \cot x - 5 = 0\)
57. \(2 \tan^2 x + 7 \tan x - 15 = 0\)
58. \(6 \sin^2 x - 7 \sin x + 2 = 0\)

Using the Quadratic Formula: In Exercises 59–62, use the Quadratic Formula to solve the equation in the interval [0, 2π]. Then use a graphing utility to approximate the angle \(x\).

59. \(12 \sin^2 x - 13 \sin x + 3 = 0\)
60. \(3 \tan^2 x + 4 \tan x - 4 = 0\)
61. \(\tan x + 3 \tan x + 1 = 0\)
62. \(4 \cos^2 x - 4 \cos x - 1 = 0\)

Using Inverse Functions: In Exercises 63–74, use inverse functions where needed to find all solutions of the equation in the interval [0, 2π).

63. \(\tan^2 x + \tan x - 12 = 0\)
64. \(\tan^2 x - \tan x - 2 = 0\)
65. \(\sec^2 x - 6 \tan x = -4\)
66. \(\sec^2 x + \tan x - 3 = 0\)
67. \(2 \sin^2 x + 5 \cos x = 4\)
68. \(2 \cos^2 x + 7 \sin x = 5\)
69. \(\cot^2 x - 9 = 0\)
70. \(\cot^2 x - 6 \cot x + 5 = 0\)
71. \(\sec^2 x - 4 \sec x = 0\)
72. \(\sec^2 x + 2 \sec x - 8 = 0\)
73. \(\csc^2 x + 3 \csc x - 4 = 0\)
74. \(\csc^2 x - 5 \csc x = 0\)

Approximating Maximum and Minimum Points: In Exercises 79–84, (a) use a graphing utility to graph the function and approximate the maximum and minimum points on the graph in the interval [0, 2π]. and (b) solve the trigonometric equation and demonstrate that its solutions are the \(x\)-coordinates of the maximum and minimum points of \(f\). (Calculus is required to find the trigonometric equation.)

Function
79. \(f(x) = \sin^2 x + \cos x\)
80. \(f(x) = \cos^2 x - \sin x\)
81. \(f(x) = \sin x + \cos x\)
82. \(f(x) = 2 \sin x + \cos 2x\)
83. \(f(x) = \sin x \cos x\)
84. \(f(x) = \sec x \tan x \cos x - 1\)

Trigonometric Equation
2 sin x cos x - sin x = 0
-2 sin x cos x - cos x = 0
cos x - sin x = 0
2 cos x - 4 sin x cos x = 0
-sin x + cos x = 0
sec x tan x + sec^2 x = 1

Number of Points of Intersection: In Exercises 85 and 86, use the graph to approximate the number of points of intersection of the graphs of \(y_1\) and \(y_2\).

85. \(y_1 = 2 \sin x\)
\(y_2 = 3x + 1\)
86. \(y_1 = 2 \sin x\)
\(y_2 = \frac{1}{3} x + 1\)

Graphical Reasoning: Consider the function \(f(x) = (\sin x)/x\) and its graph shown in the figure.

(a) What is the domain of the function?
(b) Identify any symmetry and any asymptotes of the graph.
(c) Describe the behavior of the function as \(x \to 0\).
(d) How many solutions does the equation \(\sin x / x = 0\) have in the interval \([-π, π]? Find the solutions.
2.3 Solving Trigonometric Equations

91. Sales The monthly sales $S$ (in hundreds of units) of skiing equipment at a sports store are approximated by

$$S = 58.3 + 32.5 \cos \frac{\pi t}{6}$$

where $t$ is the time (in months), with $t = 1$ corresponding to January. Determine the months in which sales exceed 7500 units.

92. Projectile Motion A baseball is hit at an angle of $\theta$ with the horizontal and with an initial velocity of $v_0 = 100$ feet per second. An outfielder catches the ball 300 feet from home plate (see figure). Find $\theta$ when the range $r$ of a projectile is given by

$$r = \frac{1}{32} v_0^2 \sin 2\theta.$$
94. Ferris Wheel

The height \( h \) (in feet) above ground of a seat on a Ferris wheel at time \( t \) (in minutes) can be modeled by

\[
h(t) = 53 + 50 \sin \left( \frac{\pi t}{16} - \frac{\pi}{2} \right)
\]

The wheel makes one revolution every 32 seconds.
The ride begins when \( t = 0 \).
(a) During the first 32 seconds of the ride, when will a person on the Ferris wheel be 53 feet above ground?
(b) When will a person be at the top of the Ferris wheel for the first time during the ride? If the ride lasts 160 seconds, then how many times will a person be at the top of the ride, and at what times?

95. Geometry

The area of a rectangle (see figure) inscribed in one arc of the graph of \( y = \cos x \) is given by

\[
A = 2x \cos x, \quad 0 < x < \frac{\pi}{2}.
\]

(a) Use a graphing utility to graph the area function and approximate the area of the largest inscribed rectangle.
(b) Determine the values of \( x \) for which \( A \geq 1 \).

96. Quadratic Approximation

Consider the function

\[
f(x) = 3 \sin(0.6x - 2).
\]
(a) Approximate the zero of the function in the interval \([0, 6]\).
(b) A quadratic approximation agreeing with \( f \) at \( x = 5 \) is

\[
g(x) = -0.45x^2 + 5.52x - 13.70.
\]
Use a graphing utility to graph \( f \) and \( g \) in the same viewing window. Describe the result.
(c) Use the Quadratic Formula to find the zeros of \( g \). Compare the zero in the interval \([0, 6]\) with the result of part (a).

97. Fixed Point

In Exercises 97 and 98, find the smallest positive fixed point of the function \( f \). [A fixed point of a function \( f \) is a real number \( c \) such that \( f(c) = c \).]

97. \( f(x) = \tan \left( \pi x / 4 \right) \)
98. \( f(x) = \cos x \)

Exploration

True or False? In Exercises 99 and 100, determine whether the statement is true or false. Justify your answer.

99. The equation \( 2 \sin 4t - 1 = 0 \) has four times the number of solutions in the interval \([0, 2\pi]\) as the equation \( 2 \sin t - 1 = 0 \).
100. If you correctly solve a trigonometric equation to the statement \( \sin x = 3.4 \), then you can finish solving the equation by using an inverse function.

101. Think About It

Explain what happens when you divide each side of the equation \( \cot x \cos^2 x = 2 \cot x \) by \( \cot x \). Is this a correct method to use when solving equations?

102. HOW DO YOU SEE IT?

Explain how to use the figure to solve the equation \( 2 \cos x - 1 = 0 \).

103. Graphical Reasoning

Use a graphing utility to confirm the solutions found in Example 6 in two different ways.
(a) Graph both sides of the equation and find the \( x \)-coordinates of the points at which the graphs intersect.

Left side: \( y = \cos x - 1 \)
Right side: \( y = \sin x \)
(b) Graph the equation \( y = \cos x + 1 - \sin x \) and find the \( x \)-intercepts of the graph. Do both methods produce the same \( x \)-values? Which method do you prefer? Explain.

104. Discussion

Explain in your own words how knowledge of algebra is important when solving trigonometric equations.

Project: Meteorology

To work an extended application analyzing the normal daily high temperatures in Phoenix and in Seattle, visit this text's website at LarsonPrecalculus.com. (Source: NOAA)
2.4 Sum and Difference Formulas

Use sum and difference formulas to evaluate trigonometric functions, verify identities, and solve trigonometric equations.

Using Sum and Difference Formulas

In this and the following section, you will study the uses of several trigonometric identities and formulas.

**Sum and Difference Formulas**

\[
\begin{align*}
\sin(u + v) &= \sin u \cos v + \cos u \sin v \\
\sin(u - v) &= \sin u \cos v - \cos u \sin v \\
\cos(u + v) &= \cos u \cos v - \sin u \sin v \\
\cos(u - v) &= \cos u \cos v + \sin u \sin v \\
\tan(u + v) &= \frac{\tan u + \tan v}{1 - \tan u \tan v} \\
\tan(u - v) &= \frac{\tan u - \tan v}{1 + \tan u \tan v}
\end{align*}
\]

For a proof of the sum and difference formulas for \(\cos(u \pm v)\) and \(\tan(u \pm v)\), see Proofs in Mathematics on page 256.

Examples 1 and 2 show how sum and difference formulas can enable you to find exact values of trigonometric functions involving sums or differences of special angles.

**Example 1** Evaluating a Trigonometric Function

Find the exact value of \(\sin \frac{\pi}{12}\).

**Solution** To find the exact value of \(\sin \frac{\pi}{12}\), use the fact that

\[
\frac{\pi}{12} = \frac{\pi}{3} - \frac{\pi}{4},
\]

Consequently, the formula for \(\sin(u - v)\) yields

\[
\sin \frac{\pi}{12} = \sin \left( \frac{\pi}{3} - \frac{\pi}{4} \right) = \sin \frac{\pi}{3} \cos \frac{\pi}{4} - \cos \frac{\pi}{3} \sin \frac{\pi}{4} = \frac{\sqrt{3}}{2} \left( \frac{\sqrt{2}}{2} \right) - \frac{1}{2} \left( \frac{\sqrt{2}}{2} \right) = \frac{\sqrt{6} - \sqrt{2}}{4}.
\]

Try checking this result on your calculator. You will find that \(\sin \frac{\pi}{12} \approx 0.259\).

**Checkpoint** Find the exact value of \(\cos \frac{\pi}{12}\).
 REMARK Another way to solve Example 2 is to use the fact that 75° = 120° - 45°, together with the formula for \cos(u-v).

**EXAMPLE 2** Evaluating a Trigonometric Function

Find the exact value of \cos 75°.

**Solution** Using the fact that 75° = 30° + 45°, together with the formula for \cos(u+v), you obtain

\[
\cos 75° = \cos(30° + 45°) = \cos 30° \cos 45° - \sin 30° \sin 45° = \frac{\sqrt{2}}{2} \left( \frac{\sqrt{2}}{2} \right) - \frac{1}{2} \left( \frac{\sqrt{2}}{2} \right) = \frac{\sqrt{6} - \sqrt{2}}{4}
\]

**Checkpoint** Audio-video solution in English & Spanish at LarsonPrecalculus.com.

Find the exact value of \sin 75°.

**EXAMPLE 3** Evaluating a Trigonometric Expression

Find the exact value of \sin(u + v) given sin u = 4/5, where 0 < u < \pi/2, and cos v = -12/13, where \pi/2 < v < \pi.

**Solution** Because sin u = 4/5 and u is in Quadrant I, cos u = 3/5, as shown in Figure 2.3. Because cos v = -12/13 and v is in Quadrant II, sin v = 5/13, as shown in Figure 2.4. You can find \sin(u + v) as follows,

\[
\sin(u + v) = \sin u \cos v + \cos u \sin v = \frac{4}{5} \left( \frac{12}{13} \right) + \frac{3}{5} \left( \frac{5}{13} \right) = \frac{33}{65}
\]

**Checkpoint** Audio-video solution in English & Spanish at LarsonPrecalculus.com.

Find the exact value of cos(u + v) given sin u = 12/13, where 0 < u < \pi/2, and cos v = -3/5, where \pi/2 < v < \pi.

**EXAMPLE 4** An Application of a Sum Formula

Write cos(arctan 1 + arccos x) as an algebraic expression.

**Solution** This expression fits the formula for \cos(u + v). Figure 2.5 shows angles u = arctan 1 and v = arccos x. So,

\[
\cos(u + v) = \cos(\arctan 1) \cos(\arccos x) - \sin(\arctan 1) \sin(\arccos x) = \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} - \sqrt{1 - x^2} = \frac{x - \sqrt{1-x^2}}{\sqrt{2}}.
\]

**Checkpoint** Audio-video solution in English & Spanish at LarsonPrecalculus.com.

Write sin(\arctan 1 + \arccos x) as an algebraic expression.
Example 5  Proving a Cofunction Identity

Use a difference formula to prove the cofunction identity \( \cos\left(\frac{\pi}{2} - x\right) = \sin x \).

Solution  Using the formula for \( \cos(u - v) \), you have

\[
\cos\left(\frac{\pi}{2} - x\right) = \cos\frac{\pi}{2} \cos x + \sin\frac{\pi}{2} \sin x
\]

\[= (0)\cos x + (1)(\sin x)\]

\[= \sin x.\]

✓ Checkpoint  Audio-video solution in English & Spanish at LarsonPrecalculus.com

Use a difference formula to prove the cofunction identity \( \sin\left(x - \frac{\pi}{2}\right) = -\cos x \).

Sum and difference formulas can be used to rewrite expressions such as

\( \sin\left(\theta + \frac{\pi}{2}\right) \) and \( \cos\left(\theta + \frac{\pi}{2}\right) \), where \( n \) is an integer

as expressions involving only \( \sin \theta \) or \( \cos \theta \). The resulting formulas are called reduction formulas.

Example 6  Deriving Reduction Formulas

Simplify each expression.

a. \( \cos\left(\theta - \frac{3\pi}{2}\right) \)

b. \( \tan(\theta + 3\pi) \)

Solution

a. Using the formula for \( \cos(u - v) \), you have

\[
\cos\left(\theta - \frac{3\pi}{2}\right) = \cos \theta \cos \frac{3\pi}{2} + \sin \theta \sin \frac{3\pi}{2}
\]

\[= (\cos \theta)(0) + (\sin \theta)(-1)\]

\[= -\sin \theta.\]

b. Using the formula for \( \tan(u + v) \), you have

\[
\tan(\theta + 3\pi) = \frac{\tan \theta + \tan 3\pi}{1 - \tan \theta \tan 3\pi}
\]

\[= \frac{\tan \theta + 0}{1 - (\tan \theta)(0)}\]

\[= \tan \theta.\]

✓ Checkpoint  Audio-video solution in English & Spanish at LarsonPrecalculus.com

Simplify each expression.

a. \( \sin\left(\frac{3\pi}{2} - \theta\right) \)  

b. \( \tan\left(\theta - \frac{\pi}{4}\right) \)
EXAMPLE 7  Solving a Trigonometric Equation

Find all solutions of \( \sin(x + (\pi/4)) - \sin(x - (\pi/4)) = -1 \) in the interval \([0, 2\pi]\).

Algebraic Solution

Using sum and difference formulas, rewrite the equation as

\[
\sin x \cos \frac{\pi}{4} + \cos x \sin \frac{\pi}{4} + \sin x \cos \frac{\pi}{4} - \cos x \sin \frac{\pi}{4} = -1
\]

\[
2 \sin x \cos \frac{\pi}{4} = -1
\]

\[
2 \sin x \left( \frac{\sqrt{2}}{2} \right) = -1
\]

\[
\sin x = -\frac{1}{\sqrt{2}}
\]

\[
\sin x = -\frac{\sqrt{2}}{2}
\]

So, the only solutions in the interval \([0, 2\pi]\) are \( x = 5\pi/4 \) and \( x = 7\pi/4 \).

✓ Checkpoint  \( \square \) Audio-video solution in English & Spanish at LarsonPrecalculus.com.

Find all solutions of \( \sin(x + (\pi/2)) + \sin(x - (3\pi/2)) = 1 \) in the interval \([0, 2\pi]\).

The next example is an application from calculus.

EXAMPLE 8  An Application from Calculus

Verify that

\[
\frac{\sin(x + h) - \sin x}{h} = (\cos x) \left( \frac{\sin h}{h} \right) - (\sin x) \left( \frac{1 - \cos h}{h} \right), \quad \text{where} \ h \neq 0.
\]

Solution  Using the formula for \( \sin(x + v) \), you have

\[
\frac{\sin(x + h) - \sin x}{h} = \frac{\sin x \cos h + \cos x \sin h - \sin x}{h}
\]

\[
= \frac{\cos x \sin h - \sin x (1 - \cos h)}{h}
\]

\[
= \left( \cos x \right) \left( \frac{\sin h}{h} \right) - \left( \sin x \right) \left( \frac{1 - \cos h}{h} \right)
\]

✓ Checkpoint  \( \square \) Audio-video solution in English & Spanish at LarsonPrecalculus.com.

Verify that

\[
\frac{\cos(x + h) - \cos x}{h} = (\cos x) \left( \frac{\cos h - 1}{h} \right) - (\sin x) \left( \frac{\sin h}{h} \right), \quad \text{where} \ h \neq 0.
\]

Summarize  (Section 2.4)

1. State the sum and difference formulas for sine, cosine, and tangent (page 235).
   For examples of using the sum and difference formulas to evaluate trigonometric functions, verify identities, and solve trigonometric equations, see Examples 1–8.
2.4 Exercises

Vocabulary: Fill in the blank.

1. \( \sin(\theta - \phi) = \) 
2. \( \cos(\theta + \phi) = \) 
3. \( \tan(\theta + \phi) = \) 
4. \( \sin(\theta + \phi) = \) 
5. \( \cos(\theta - \phi) = \) 
6. \( \tan(\theta - \phi) = \) 

Skills and Applications

Evaluating Trigonometric Expressions In Exercises 7–10, find the exact value of each expression.

7. (a) \( \cos\left(\frac{\pi}{4} + \frac{\pi}{3}\right) \) 
   (b) \( \cos\frac{\pi}{4} + \cos\frac{\pi}{3} \)

8. (a) \( \sin\left(\frac{7\pi}{6} - \frac{\pi}{3}\right) \) 
   (b) \( \sin\frac{7\pi}{6} - \sin\frac{\pi}{3} \)

9. (a) \( \sin(135^\circ - 30^\circ) \) 
   (b) \( \sin 135^\circ - \cos 30^\circ \)

10. (a) \( \cos(120^\circ + 45^\circ) \) 
    (b) \( \cos 120^\circ + \cos 45^\circ \)

Evaluating Trigonometric Functions In Exercises 11–26, find the exact values of the sine, cosine, and tangent of the angle.

11. \( \frac{11\pi}{12} = \) 
12. \( \frac{7\pi}{12} = \)

13. \( \frac{17\pi}{12} = \) 
14. \( \frac{14\pi}{12} = \)

15. \( 105^\circ = 60^\circ + 45^\circ \) 
16. \( 165^\circ = 135^\circ + 30^\circ \)

17. \( 195^\circ = 225^\circ - 30^\circ \) 
18. \( 255^\circ = 300^\circ - 45^\circ \)

19. \( \frac{13\pi}{12} = \) 
20. \( \frac{7\pi}{12} = \)

21. \( \frac{-13\pi}{12} = \) 
22. \( \frac{5\pi}{12} = \)

23. \( 285^\circ = \) 
24. \( -105^\circ = \)

25. \( -165^\circ = \) 
26. \( 15^\circ = \)

Rewriting a Trigonometric Expression In Exercises 27–34, write the expression as the sine, cosine, or tangent of an angle.

27. \( \sin 3 \cos 1.2 - \cos 3 \sin 1.2 \)
28. \( \cos \left(\frac{\pi}{7} \cos \frac{\pi}{5} - \sin \frac{\pi}{7} \sin \frac{\pi}{5}\right) \)
29. \( \sin 60^\circ \cos 15^\circ + \cos 60^\circ \sin 15^\circ \)
30. \( \cos 130^\circ \cos 40^\circ - \sin 130^\circ \sin 40^\circ \)
   \( \tan 45^\circ - \tan 30^\circ \)
\( 1 - \tan 45^\circ \tan 30^\circ \)
\( \tan 140^\circ - \tan 60^\circ \)
\( 1 + \tan 140^\circ \tan 60^\circ \)

33. \( \cos 3 \cos 2y + \sin 3 \sin 2y \frac{\tan 2x + \tan y}{1 - \tan 2x \tan y} \)
34. \( \frac{\tan 2x + \tan y}{1 - \tan 2x \tan y} \)

Evaluating a Trigonometric Expression In Exercises 35–40, find the exact value of the expression.

35. \( \sin \frac{\pi}{12} \cos \frac{\pi}{4} + \cos \frac{\pi}{12} \sin \frac{\pi}{4} \)
36. \( \cos \frac{\pi}{12} \cos \frac{3\pi}{16} + \cos \frac{\pi}{12} \sin \frac{3\pi}{16} \)
37. \( \cos 120^\circ \cos 60^\circ - \cos 120^\circ \sin 60^\circ \)
38. \( \cos 120^\circ \cos 30^\circ \cot \sin 120^\circ \sin 30^\circ \)
39. \( \frac{\tan(5\pi/6) - \tan(\pi/6)}{1 + \tan(5\pi/6) \tan(\pi/6)} \)
40. \( \frac{\tan 25^\circ - \tan 110^\circ}{1 - \tan 25^\circ \tan 110^\circ} \)

Evaluating a Trigonometric Expression In Exercises 41–46, find the exact value of the trigonometric expression given that \( \sin \theta = \frac{5}{13} \) and \( \cos \phi = -\frac{3}{5} \) (Both \( \theta \) and \( \phi \) are in Quadrant II.)

41. \( \sin(\theta + \phi) \)
42. \( \cos(\theta - \phi) \)
43. \( \tan(\theta + \phi) \)
44. \( \csc(\theta - \phi) \)
45. \( \sec(\theta - \phi) \)
46. \( \cot(\theta + \phi) \)

Evaluating a Trigonometric Expression In Exercises 47–52, find the exact value of the trigonometric expression given that \( \sin \theta = -\frac{\sqrt{2}}{2} \) and \( \cos \phi = -\frac{\sqrt{3}}{2} \) (Both \( \theta \) and \( \phi \) are in Quadrant III.)

47. \( \cos(\theta - \phi) \)
48. \( \sin(\theta + \phi) \)
49. \( \tan(\theta - \phi) \)
50. \( \cot(\theta + \phi) \)
51. \( \sec(\theta - \phi) \)
52. \( \csc(\theta + \phi) \)

An Application of a Sum or Difference Formula In Exercises 53–56, write the trigonometric expression as an algebraic expression.

53. \( \sin(\arcsin x + \arccos x) \)
54. \( \sin(\arctan 2x - \arccos x) \)
55. \( \cos(\arccos x + \arcsin x) \)
56. \( \cos(\arccos x - \arctan x) \)
Chapter 2
Section 2.1 (page 215)

1. \( \tan x = \frac{1}{2} \)  
3. \( \cot x \)  
5. \( \cot x \)

7. \( \sin x = \frac{1}{2} \)  
9. \( \cos x = \frac{3}{5} \)  
11. \( \sin x = \frac{\sqrt{15}}{4} \)  
13. \( \sin \theta = -1 \)  
15. \( \cos x = \frac{1}{4} \)  
17. \( \tan x = \frac{\sqrt{15}}{4} \)  
19. \( \csc x = \frac{\sqrt{15}}{3} \)  
21. \( \sin x \)  
23. \( \sec x + 1 \)  
25. \( \sin x \)

27. \( \sec (\tan x + 1) \)  
29. \( (\sin x - 1)(\sin x + 2) \)  
31. \( (\sec x - 1)(\sec x + 2) \)  
33. \( \frac{1}{2} + 2 \sin x \cos x \)  
35. \( \cos \theta \)  
37. \( \cos^2 \phi \)  
39. \( \sin^2 x = 1 \)  
41. \( \tan x \)

43. \( \sec \beta \)  
45. \( 2 \sec^2 x \)  
47. \( -\cot x \)  
49. \( 1 + \cos x \)

51. \( \csc x = 3 \)  
53. \( \sin \theta = 0 \)  
55. \( 2 \tan \theta \)

57. \( 3 \cos \theta = 3; \sin \theta = 0; \cos \theta = 1 \)  
59. \( 0 \leq \theta \leq \pi \)  
61. \( \mu = \tan \theta \)  
63. \( \frac{7\pi}{4} \)

65. \( \infty, 0 \)  
67. Not an identity because \( \frac{\sin k\theta}{\cos k\theta} = \tan k\theta \)

69. \( \cot \) or \( \sec x \)

71. \( \cos \theta = \pm \sqrt{1 - \sin^2 \theta} \)  
\( \tan \theta = \pm \frac{x}{1 - \sin^2 \theta} \)  
\( \cot \theta = \pm \frac{1}{\sin^2 \theta} \)  
\( \sin \theta = \pm \sqrt{1 - \cos^2 \theta} \)  
\( \csc \theta = \frac{1}{\cos \theta} \)

Section 2.2 (page 222)

1. Identity  
3. \( \tan x \)  
5. \( \cos x \)  
7. \( -\csc x \)

9–49. Answers will vary.

51. In the first line, \( \cot(x) \) is substituted for \( \cot(-x) \), which is incorrect; \( \cot(-x) = -\cot(x) \).