Practice Test One

(1) Below is a graph of the function $f(x)$. Give the requested information about $f(x)$.

(1a) $f(-1)$   
(1b) $f(0)$   
(1c) $f(1)$   
(1d) $\lim_{x \to -1} f(x)$   
(1e) $\lim_{x \to 0} f(x)$   
(1f) $\lim_{x \to -1^-} f(x)$   
(1g) $\lim_{x \to -1^+} f(x)$   
(1h) $\lim_{x \to 1^-} f(x)$   
(1i) $\lim_{x \to 2^-} f(x)$   
(1j) $\lim_{x \to 2^+} f(x)$

2) Calculate the following limits or indicate that the limit Does Not Exist (DNE).

(2a) $\lim_{x \to 2} \frac{x^2 - 4}{x - 2}$

(2b) $\lim_{x \to 0} \frac{x^3 - 4x}{4x}$

(2c) $\lim_{x \to 1} \frac{\sqrt{x} - 1}{x - 1}$

(2d) $\lim_{x \to 2} \frac{x^2 + 4}{x - 2}$
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3) Let \( f(x) \) be the piecewise function defined be the formula below.

\[
f(x) = \begin{cases} 
\frac{\sin(x)}{x} & \text{if } x < 0 \\
x^2 + 1 & \text{if } 0 \leq x \leq 1 \\
\frac{x^2 - 10x + 9}{x^2 - 4x + 3} & \text{if } 1 < x < 3 \\
-6 & \text{if } x = 3 \\
\frac{x - 9}{x^2 - 9} & \text{if } 3 < x 
\end{cases}
\]

State all \( x \) values at which \( f(x) \) is continuous. State all \( x \) values at which \( f(x) \) is discontinuous. For each real number \( x \), explain why \( f(x) \) is or is not continuous at \( x \).

4) Sketch the graph of the following function \( g(x) \). Label any horizontal asymptotes, vertical asymptotes and holes in the graph, as well as any \( x \) or \( y \) intercepts.

\[
g(x) = \frac{2x^2 - 13x + 15}{x^2 - 4x - 5}.
\]

5) Let

\[
f(x) = 3x^2 + 2x + 1.
\]

5a) Compute the difference quotient for \( f(x) \) at \( x = 1 \), \( \frac{f(1+h) - f(1)}{h} \).

5b) Simplify the difference quotient to remove the division by \( h \).

5c) Take the limit of your answer to (5b), to calculate \( f'(1) \).

5d) Use the power rule to calculate \( f'(x) \)

5e) Evaluate your answer to (5d) at \( x = 1 \), to calculate \( f'(1) \).

5f) Calculate the equation of the tangent line to the graph of \( f(x) \) at \( x = 1 \).
(1) Below is a graph of the function $f(x)$. Give the requested information about $f(x)$.

<table>
<thead>
<tr>
<th>$f(-1)$</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f(0)$</td>
<td>0</td>
</tr>
<tr>
<td>$f(1)$</td>
<td>3</td>
</tr>
<tr>
<td>$\lim_{x\to-1} f(x)$</td>
<td>1</td>
</tr>
<tr>
<td>$\lim_{x\to 0} f(x)$</td>
<td>$-\infty$</td>
</tr>
</tbody>
</table>

(1f) $\lim_{x\to-1} f(x)$

(1g) $\lim_{x\to 1^+} f(x)$

(1h) $\lim_{x\to 1} f(x)$

(1i) $\lim_{x\to 2^-} f(x)$

(1j) $\lim_{x\to 2^+} f(x)$

2) Calculate the following limits or indicate that the limit Does Not Exist (DNE).

(2a) $\lim_{x\to 2} \frac{x^2-4}{x-2} = \lim_{x\to 2} \frac{(x-2)(x+2)}{x-2} = \lim_{x\to 2} x + 2 = 2 + 2 = 4$.

(2b) $\lim_{x\to 0} \frac{x^3-4x}{4x} = \lim_{x\to 0} \frac{x(x^2-4)}{4x} = \lim_{x\to 0} \frac{x^2-4}{4} = \frac{-4}{4} = -1$.

(2c) $\lim_{x\to 1} \frac{\sqrt{x}-1}{x-1} = \lim_{x\to 1} \frac{\sqrt{x}-1}{(\sqrt{x}+1)(\sqrt{x}-1)} = \lim_{x\to 1} \frac{\sqrt{x}-1}{\sqrt{x}+1} = \lim_{x\to 1} \frac{1}{1+1} = \frac{1}{2}$.

(2d) $\lim_{x\to 2} \frac{x^2+4}{x-2}$ DNE (as $x$ approaches 2, the function values become infinite.)

4) Sketch the graph of the function $g(x) = \frac{2x^2-13x+15}{x^2-4x-5}$. Label any horizontal asymptotes, vertical asymptotes and holes in the graph, as well as any $x$ or $y$ intercepts.

The function $g(x)$ is not defined at $x = -1$ or $x = 5$. Indeed,

$$g(x) = \frac{2x^2-13x+15}{x^2-4x-5} = \frac{(2x-3)(x-5)}{(x-5)(x+1)} = \frac{2x-3}{x+1}.$$ 

The $y$ intercept is the ordered pair $(0, g(0)) = (0, -3)$. The $x$ intercepts occur where the value of $g(x)$ is zero. That is, at $\left(\frac{5}{2}, 0\right)$. Now, $\lim_{x\to 5} g(x) = \lim_{x\to 5} \frac{2x-3}{x+1} = \frac{7}{6}$ and the graph of $g$ has hole at $(5, \frac{7}{6})$. There is a vertical asymptote at $x = -1$. Since $\lim_{x\to -1^-} g(x) = +\infty$ and $\lim_{x\to -1^+} g(x) = -\infty$, the graph climbs up the vertical asymptote on the left and fall down this asymptote on the right. There is a horizontal asymptote at $y = 2$. The graph of $g(x)$ is above the horizontal line $y = 2$ on the left of the vertical line $x = -1$ and below $y = 2$ on the right of $x = -1$. The graph of $g(x)$ looks like the graph of $\frac{1}{x}$, shifted left one unit and up 2 units.

3) Let $f(x)$ be the piece-wise function define be the formula below.

$$f(x) = \begin{cases} 
\sin(x) \quad &\text{if } x < 0 \\
x^2 + 1 \quad &\text{if } 0 \leq x \leq 1 \\
\frac{x^2-10x+9}{x^2-4x+3} \quad &\text{if } 1 < x < 3 \\
-6 \quad &\text{if } x = 3 \\
\frac{x-9}{x^2-9} \quad &\text{if } 3 < x 
\end{cases}$$

State all $x$ values at which $f(x)$ is continuous. State all $x$ values at which $f(x)$ is discontinuous. For each real number $x$, Explain why $f(x)$ is or is not continuous at $x$. 1
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(3) SOLUTION: We look at the $x$ values where the definition of $f(x)$ changes. Now, $\lim_{x \to 0^-} x \sin(x) = 0$ and $\lim_{x \to 0^+} x^2 + 1 = 1$. Since the left and right limits are equal, $\lim_{x \to 0} f(x) = 1$. Also, $f(0) = 0^2 + 1 = 1$. Since the function value equals the limit at $x = 0$, $f(x)$ is continuous at 0.

Next $\lim_{x \to 1^-} f(x) = 1^2 + 1 = 2$ and

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} \frac{(x-1)(x-9)}{(x-3)} = \lim_{x \to 1^+} \frac{x-9}{x-3} = \frac{-8}{-2} = 4.$$ 

So, the limit of $f(x)$ as $x$ approaches 1 does not exist. So, $f(x)$ is discontinuous at $x = 1$.

Finally,

$$\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} \frac{(x-1)(x-9)}{(x-3)} = \lim_{x \to 3^-} \frac{x-9}{x-3} = \infty$$

and

$$\lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} \frac{x-9}{x-3} = -\infty.$$ 

Since the limit of $f(x)$ as $x$ approaches 3 does not exist, $f(x)$ is discontinuous at $x = 3$.

Because the formulas defining $f(x)$ do not change or become undefined, we know from the behavior of polynomials, rational functions and trigonometric functions that $f$ is continuous at all other $x$ values. In summary, $f(x)$ is continuous at all real values of $x$ except $x = 1$ and $x = 3$.

5) Let

$$f(x) = 3x^2 + 2x + 1.$$ 

(5a) Compute the difference quotient for $f(x)$ at $x = 1$, $\frac{f(1+h) - f(1)}{h}$.

$$\frac{f(1+h) - f(1)}{h} = \frac{[3(1+h)^2 + 2(1+h) + 1] - [3(1)^2 + 2(1) + 1]}{h}.$$ 

(5b) Simplify the difference quotient to remove the division by $h$.

$$\frac{[3(1 + 2h + h^2) + 2(1 + h) + 1] - [3 + 2 + 1]}{h} = \frac{3 + 6h + 3h^2 + 2 + 2h + 1 - 6}{h} = \frac{8h + 3h^2}{h}.$$ 

So, the difference quotient simplifies as $\frac{h(8+3h)}{h} = 8 + 3h$.

(5c) Take the limit of your answer to (5b), to calculate $f'(1)$. $f'(1) = \lim_{h \to 0} 8 + 3h = 8$.

(5d) Use the power rule to calculate $f'(x)$.

The power rule applied to $f(x) = 3x^2 + 2x + 1$ gives us $f'(x) = 6x + 2$.

(5e) Evaluate your answer to (5d) at $x = 1$, to calculate $f'(1)$. $f'(1) = 6(1) + 2 = 8$.

(5f) Calculate the equation of the tangent line to the graph of $f(x)$ at $x = 1$.

The point of tangency is $(1, f(1)) = (1, 6)$ and by the above calculations this tangent line has slope 8. The equation of the line is $y - 6 = 8(x - 1)$ or $y = 8x - 2$. 
