Chapter 2 - weighted voting

Weighted Voting Only Vote yes/no on motion
Players have multiple votes

Form of weighted voting system is
\[ [q: w_1, w_2, \ldots, w_n] \]
Convention: \( w_1 \geq w_2 \geq \cdots \geq w_n \). Player \( k \) has \( w_k \) votes.

Total Votes \( V = w_1 + w_2 + \cdots + w_n \)

Valid Quota \( q \) satisfies
\[
\frac{V}{2} < q \leq V
\]
q = quota motion to pass requires 10 or more votes (quota)

Player 1 has 8 votes, Player 2 has 4 votes, Player 3 has 3 votes, Player 4 has 2 votes

Total votes $V = 8 + 4 + 3 + 2 = 17$. MEET QUOTA REQUIRES $\frac{10}{17}$

Form for weighted voting system is $[q: w_1, w_2, \ldots w_n]$ $w_1 \geq w_2 \geq \cdots \geq w_n$ and $q = \text{quota}$

$q = \text{yes votes need to pass}$

Restriction on valid quota $V = \text{total number of votes}$

$$\frac{V}{2} < q \leq V$$
If $q \leq \frac{V}{2}$ you get ANARCHY - both Yes and No can meet quota.
If $q > V$ you get GRIDLOCK - can NEVER meet quota.

Winning coalitions = Group of players with enough votes to meet or exceed the quota.

Losing coalition = Group of player whose votes total less than the quota.
Winning coalitions:

\{P_1, P_2, P_3, P_4\} (17)
\{P_1, P_2, P_3\} (15),
\{P_1, P_2, P_4\} (14),
\{P_1, P_3, P_4\} (13),
\{P_1, P_2\} (12),
\{P_1, P_3\} (11), \{P_1, P_4\} (10)

\{P_2, P_3, P_4\} Value = 4 + 3 + 2 = 9

Losing coalition
Notice Player 1 is in every winning coalition. Any Player that is every winning coalition is said to have **VETO POWER**.

This happens when the sum of all other players is less than $q$.

Losing coalition: $\{P_2, P_3, P_4\}$ (9)

Player 1 is a **dictator** if Player 1 has enough votes to meet the quota alone.
Example: [5: 6, 2, 1] Here a motion passes if Player 1 votes yes and fails if Player 1 votes no.

Players 2 and 3 are **dummies** they are never needed to form a winning coalition.
4 players,
number of 4 member coalitions = 1
Number of 3 member coalitions = 4 = C(4,3) =
(just choose 1 player to leave out)
2 member coalitions = C(4,2)

Recall $C(n,k) =$ Choose set of $k$ objects out of $n$ objects

\[ C(n, k) = \frac{n!}{k! (n - k)!} \]

Example
\[ C(7,5) = \frac{7!}{5! 2!} = \frac{5040}{(120)(2)} = 21 \]

If there are 7 players then there are 21 coalitions with 5 members.

[51: 49, 48, 3] Republican = 49, Democrat = 48, Indepen = 3

Find all winning coalitions:
\{R,D,I\}(100), \{R,D\}(97), \{R,I\}(52), \{D,I\}(51)

Every group, R,D and I are in 3 of the 4 winning coalitions. Each has 33.3% of the power
Winning coalitions = groups total 51 or more
R=49, D = 48, a = Ind 1, b = Ind 2, c = independent 3
{R,D,a,b,c}(100) = Grand coalition = unanimous
{R,D,a,b}(99) {R,D, a}(98) {R,D, b}(98) {R,D, c}(98)
{R,D,a,c}(99) {R,a,b}(51) {R,a,c}(51) {R,c,b}(51)
{R,D,b,c}(99) {R,D}(97)
{R,a,b,c}(52)
{D,a,b,c}(51)
We had R = D = I = 33.3% of power when I= {a,b,c} vote same

Now R has more power than D. D has more than an indep.
Compare power of a, b and c: The same power for a,b,c
A player in a winning coalition is "critical" if when that player is removed, the coalition no longer meets the quota.

<table>
<thead>
<tr>
<th>Player</th>
<th>Critical Usages</th>
<th>Percent of power</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>7</td>
<td>7/10 = 70%</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>1/10 = 10%</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>1/10 = 10%</td>
</tr>
<tr>
<td>4</td>
<td>1</td>
<td>1/10 = 10%</td>
</tr>
<tr>
<td>Total</td>
<td>10</td>
<td></td>
</tr>
</tbody>
</table>

Banzhaf Power distribution

Note that player 1 has veto power since they are in every winning coalition.

- It is possible for more than one player to have veto power.
A player in a winning coalition is “critical” if when that player is removed, the coalition no longer meets the quota.

**Banzhaf power distribution:**

1. List all winning coalitions (need counting methods to check if you have them all)
2. Within each winning coalition, identify all critical players (tedious)
3. Let $B_k$ = number of critical uses of player $k$
   Let $T$ be the total number of critical usages
4. The power index of player $k$ is $\frac{B_k}{T}$ (their critical usages divided by the total critical usages)
[4: 3, 2, 1] quota=4  \( P_1 \) has 3 votes, \( P_2 \) has 2 votes, \( P_3 \) has 1
Notice \( \{P_2, P_3\} \) has value \( 2 + 1 = 3 < 4 = q \). IT IS LOSING

\[
\begin{array}{|c|c|c|c|}
\hline
\text{Winning Coalitions} & P_1 & P_2 & P_3 \\
\hline
\{P_1, P_2, P_3\} (6) & + & - & - \\
\{P_1, P_2\} (5) & + & + & - \\
\{P_1, P_3\} (4) & + & - & + \\
\hline
\text{Total critical usages:} & 3 & 1 & 1 \\
\hline
\text{Power index:} & 3/5=60\% & 1/5=20\% & 1/5=20\% \\
\hline
\end{array}
\]

Player 1 has veto power (they are in every winning coalition)
[6: 4, 3, 2, 1] quota=6

<table>
<thead>
<tr>
<th>Winning Coalitions</th>
<th>$P_1$</th>
<th>$P_2$</th>
<th>$P_3$</th>
<th>$P_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>${P_1, P_2, P_3, P_4}$ (10)</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>${P_1, P_2, P_3}$ (9)</td>
<td>+</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>${P_1, P_2, P_4}$ (8)</td>
<td>+</td>
<td>+</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>${P_1, P_3, P_4}$ (7)</td>
<td>+</td>
<td>-</td>
<td>+</td>
<td>-</td>
</tr>
<tr>
<td>${P_2, P_3, P_4}$ (6)</td>
<td>-</td>
<td>+</td>
<td>+</td>
<td>+</td>
</tr>
<tr>
<td>${P_1, P_2}$ (7)</td>
<td>+</td>
<td>+</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>${P_1, P_3}$ (6)</td>
<td>+</td>
<td>-</td>
<td>+</td>
<td>-</td>
</tr>
<tr>
<td>Total critical usages:</td>
<td>5</td>
<td>3</td>
<td>3</td>
<td>1</td>
</tr>
<tr>
<td>Power index:</td>
<td>$\frac{5}{12} = 42%$</td>
<td>$\frac{3}{12} = 25%$</td>
<td>$\frac{3}{12} = 25%$</td>
<td>$\frac{1}{12} = 8%$</td>
</tr>
</tbody>
</table>

Total critical usages = 12