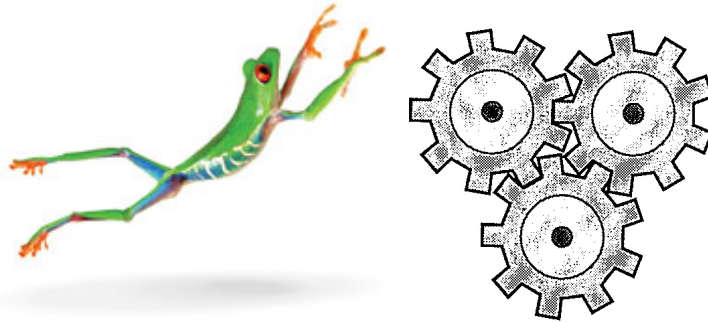


## Parity



**Problem 1.** Eleven gears are connected in a circle. Can they all rotate?

No, as two adjacent gears rotate in opposite directions.

**Problem 2.** Seven integers are written around a circle. Show that you can find two numbers next to each other whose sum is even.

If there either two even numbers or two odd numbers next to each other, we are done. If not, even and odd numbers have to alternate, which is impossible as we have 7 (odd!) numbers total around the circle.

**Problem 3.** Let  $m$  and  $n$  be integers. Show that  $mn(m + n)$  is even.

If either  $m$  or  $n$  is even, our number  $mn(m + n)$  is also even. If both  $m$  and  $n$  are odd, their sum is even and  $mn(m + n)$  is also even.

**Problem 4.** A cashier has only pennies, nickels, and quarters. When I asked him to break a \$1 bill he gave me 25 coins. Do you think the cashier made a counting mistake?

Yes, he made a mistake. If you add up an odd (25) number of odd numbers (1,5,25) you get an odd number, so you cannot get 100.

**Problem 5.** You start with 10 sheets of paper. Cut a few of them into 7 pieces, and a few of them into 5 pieces. Then you cut some of the obtained sheets into 7 or 5 pieces again, and so on. Can you get 2007 sheets of paper this way?

No. You have an even number of sheets. If you cut one of them into 7 pieces, there will be 6 more sheets, so the total number is even again. Same with cutting into 5 pieces. Does not matter how many times you repeat this, you always get an even number of pieces.

**Problem 6.** Is it possible to connect 7 computers with cables so that every computer is connected to exactly three other computers?

If this were possible, the number of cables would have been  $(3 \times 7)/2$ , which is not an integer.

**Problem 7.** Seven robins are sitting on seven pine trees planted in a row, one robin on each tree. The robins are playing a game. If one of them flies a few trees in one direction, another robin flies the same number of trees but in the opposite direction. Can they all end up on the same tree? What if you have six robins on six trees?

For the first question, yes, it is possible. For example, the robin from 1st tree flies to the 4th tree. In response, the robin from 7th tree flies to the 4th tree. Then the robins from 2nd and 6th trees fly to the 4th and the robins from 3rd and 5th trees also fly to the 4th, done.

If we have 6 robins on 6 trees the answer is no. Enumerate the trees from 1 to 6. Mark each robin with the number of the tree it's at at each time (these numbers change when robins fly from one tree to another). The sum of all robin's numbers never changes. If the number of one robin goes up there is another robin whose number goes down by the same amount. Hence the sum of all robin's numbers is  $1 + 2 + 3 + 4 + 5 + 6 = 21$ . If all robins were to end up on the  $k$ th tree, the sum of their numbers would be  $6k$ . We get  $6k = 21$ , no integer solutions.

**Problem 8.** Bob and Phil play domino. Before the game started Bob's little brother Dan stole one of the domino tiles. When the game was over all the tiles were lined up on the table with 3 dots on one end and 5 dots on the other. Which tile did Dan steal?

There are 8 half-dominos with 5 dots on them. They come in pairs inside the line hence the 5 dots that we see on one side have no pair. Same with the 3 dots. Hence the stolen tile is 3-5.

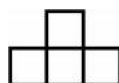
**Problem 9.** Is it possible to draw a closed (starts and ends at the same point) 6-edge broken line such that every edge intersects exactly one other edge? Is it possible to draw such a 7-edge broken line?

Yes to the first question (draw it!), no the second. The reason is that it is impossible to break 7 edges into pairs.

**Problem 10.** Can you tile a chess board with a corner square cut out with  $2 \times 1$  domino tiles? What if two opposite squares are cut out?

No to first question. You cannot tile 63 squares with 2-square tiles. No to the second question. Look at the colors of the remaining squares. You have 30 black squares and 32 white squares while each tile covers 1 black square and 1 white square.

**Problem 11.** Can you tile a  $10 \times 10$  "chess" board with 4-square tiles of the shape



No. Such a piece either covers 3 black squares and one white square or the other way around. Assume that we have  $k$  squares of the first kind and  $l$  squares of the second kind. Then

$$\begin{aligned} 3k + l &= 50 \\ k + 3l &= 50, \end{aligned}$$

and this system has no integer solutions.

**Problem 12.** Three frogs are sitting in three corners of a square. They play leapfrog taking turns leaping over each other. If, say, frog A leaps over frog B then frog B is exactly in the middle between the positions of frog A before the leap and after the leap. Can one of the frogs at some point jump to the fourth corner of the square?

Let one of the frogs be at the origin and the other two have coordinates  $(0, 1)$  and  $(1, 1)$ . If a frog leaps over some point (that is, reflects itself with respect to that point) its  $x$  and  $y$  coordinates change by an even number. Hence the first frog will always have both coordinates even, the second will have first coordinate even and the second odd, and the third will have both coordinates odd. We proved that they will never get to the fourth vertex where the first coordinate is odd and the second is even.