Fundamental Concepts of Algebra
Spring 2008

Homework 10
Due Thursday April 3rd
Exam 2 practice

1. Find $\gcd(7404621, 73122)$ and write it as a linear combination of 7404621 and 73122.

$$\begin{aligned}
7404621 &= 101 \cdot 73122 + 19299\\
73122 &= 3 \cdot 19,299 + 15225\\
19299 &= 1 \cdot 15225 + 4074\\
15225 &= 3 \cdot 4074 + 3003\\
4074 &= 1 \cdot 3003 + 1071\\
3003 &= 2 \cdot 1071 + 861\\
1071 &= 1 \cdot 861 + 210\\
861 &= 4 \cdot 210 + 21\\
210 &= 10 \cdot 21 + 0.
\end{aligned}$$

So the gcd is 21. To write this as a linear combination, we work our way up from the bottom:

$$\begin{aligned}
21 &= 861 - 4 \cdot 210\\
&= 861 - 4(1071 - 861) = 5 \cdot 861 - 4 \cdot 1071\\
&= 5(3003 - 2 \cdot 1071) - 4 \cdot 1071 = 5 \cdot 3002 - 14 \cdot 1071\\
&= 5 \cdot 3003 - 14(4074 - 3003) = 19 \cdot 3003 - 14 \cdot 4074\\
&= 19(15225 - 3 \cdot 4074) - 14 \cdot 4704 = 19 \cdot 15225 - 71 \cdot 4704\\
&= 19 \cdot 15225 - 71(19299 - 15225) = 90 \cdot 15225 - 71 \cdot 19299\\
&= 90(73122 - 3 \cdot 19299) - 71 \cdot 19299 = 90 \cdot 73122 - 341 \cdot 19299\\
&= 90 \cdot 73122 - 341(7404612 - 101 \cdot 73122) = 34531 \cdot 73122 - 341 \cdot 7404621.
\end{aligned}$$

2. Use induction to show that

$$5^0 + 5^1 + 5^2 + 5^3 + \cdots + 5^n = \frac{5^{n+1} - 1}{4}.$$

Basis step: Check this formula for $n = 0$: Indeed, $5^0 = 1 = \frac{5^{0+1}-1}{4}$.
Inductive step: Suppose that the formula holds true for $n = k$:

$$5^0 + 5^1 + 5^2 + 5^3 + \cdots + 5^k = \frac{5^{k+1} - 1}{4}, \tag{0.1}$$

and show that the formula holds for $n = k + 1$:

$$5^0 + 5^1 + 5^2 + 5^3 + \cdots + 5^k + 5^{k+1} = \frac{5^{k+2} - 1}{4}$$

Add 5^{k+1} to the both sides of (0.1). Get

$$\begin{aligned} 5^0+5^1+5^2+5^3+\cdots+5^k+5^{k+1} &= \frac{5^{k+1}-1}{4}+5^{k+1} \\ &= \frac{5^{k+1}+4\cdot 5^{k+1}-1}{4} \\ &= \frac{5\cdot 5^{k+1}-1}{4} \\ &= \frac{5^{k+2}-1}{4} \end{aligned}$$

which proves the desired formula.

3. Find a value of x that satisfies both congruences

$$\begin{cases} 3x \equiv 4 \bmod 5 \\ 2x \equiv 3 \bmod 9 \end{cases}$$

Answer: For example, $x = 33$.

4. Given a four-digit number $\overline{abcd}$ such that $\overline{ad} - \overline{bc}$ is divisible by 11, show that the number itself is also divisible by 11.

By the divisibility test, an integer is congruent to the alternating sum of its digits modulo eleven. Hence

$$\overline{abcd} \equiv (a-b+c-d) \bmod 11,$$

$$\overline{ad} \equiv (a-d) \bmod 11,$$

and

$$\overline{bc} \equiv (b-c) \bmod 11.$$

Subtracting the last congruence from the previous one we get

$$\overline{ad} - \overline{bc} \equiv (a-d)-(b-c) \bmod 11 \equiv (a-b+c-d) \bmod 11$$

and finally compare this to the first congruence and get

$$\overline{abcd} \equiv (\overline{ad} - \overline{bc}) \bmod 11$$

Hence whenever $\overline{abcd}$ is divisible by 11, $\overline{ad} - \overline{bc}$ is divisible as well.

5. What can a prime number be equal to modulo 18? *Answer:* 1, 2, 3, 5, 7, 11, 13, 17
6. Let n be an integer. What n^2+1 can be equal to modulo 3?

Answer: 1, 2

7. Given that $a \equiv b \bmod m$ and $c \equiv d \bmod m$ show that $a - c \equiv b - d \bmod m$.

We are given

$$a - b = km$$

$$c - d = lm$$

for some integers k and l. Subtract second equality form the first get

$$(a-b)-(c-d) = (a-c)-(b-d) = km - lm = m(k-l).$$

Hence $m \mid (a-c)-(b-d)$ which implies

$$a - c \equiv (b-d) \bmod m,$$

8. Prove that
$$16^{99} + 33^{100}$$
is divisible by 17.

We reduce this modulo 17. First,
$$16 \equiv -1 \bmod 17,$$
hence
$$16^{99} \equiv (-1)^{99} \bmod 17 \equiv -1 \bmod 17.$$
Next,
$$33 \equiv -1 \bmod 17,$$
hence
$$33^{100} \equiv (-1)^{100} \bmod 17 \equiv 1 \bmod 17.$$
Adding up the two obtained congruences we get
$$16^{99} + 33^{100} \equiv -1 + 1 \bmod 17 \equiv 0 \bmod 17,$$
and we get $17 \mid 16^{99} + 33^{100}$.

9. Reduce 5^{100} modulo 6.
$$5 \equiv -1 \bmod 6,$$
hence
$$5^{100} \equiv (-1)^{100} \bmod 6 \equiv 1 \bmod 6.$$

10. Find all the integer solutions of
$$x^2 = 14 + y^2$$

This is equivalent to
$$(x - y)(x + y) = 14$$
Since we are only looking for integer solutions we need to look at all possible integer factorization with factors (order matters). For example, $14 = 2 \cdot 7$ corresponds to
$$\begin{cases} x - y = 2 \\ x + y = 7 \end{cases}$$
and this system has no integer solutions. Going through all eight cases we see that there are no integer solutions.