Fundamental Concepts of Algebra
Spring 2008

Notes 2/7/08

We first talk about inequalities involving absolute values. In the homework you have a couple of inequalities of the form

$$|f(x)| \leq |g(x)|$$

Both sides are positive, so we can square the inequality to get an equivalent one

$$f^2(x) \leq g^2(x) \tag{0.1}$$

This helped us get rid of the absolute values. Use this to solve 5(a) on the homework. If you do the same thing with 5(b) you will get an equation of 4th degree, which you don't to deal with. You can notice instead that (0.1) is equivalent to

$$\begin{aligned} f^2(x) - g^2(x) &\leq 0 \\ (f(x) - g(x))\,(f(x) + g(x)) &\leq 0 \end{aligned}$$

Use this to solve 5(b).

Now consider an inequality of the form

$$|f(x)| < g(x)$$

First, in order for x to belong to the solution set we need $g(x) > 0$. Otherwise, we have something positive on the left-hand side and something non-positve on the right, which is impossible. Hence the original inequality is equivalent to

$$\begin{cases} g(x) > 0 \\ -g(x) < f(x) < g(x) \end{cases}$$

Notice that $g(x) > 0$ is redundant here as second line implies

$$\begin{aligned} -g(x) &< g(x) \\ 2g(x) &> 0 \\ g(x) &> 0 \end{aligned}$$

and the original inequality is equivalent to

$$-g(x) < f(x) < g(x)$$

Use this in to solve problem 8.

We will stop talking about inequalities now. I will keep putting inequalities on the homework though and we will occasionally discuss methods needed to deal with those homework questions.

Number Theory - Integers and Divisibility

We will use Ulrike's notes, Chapter 9 (http://www.math.kent.edu/ soprunova/34001/vor.pdf). Don't print the notes yet, I am trying to get my hands on a newer version.

Before we go into any theory, we will solve a few problems to develop intuition about integers and divisibility as well as review some concepts you learned in high-school.

Example 1. Represent each of the following numbers as a product of primes: 111, 1111, 111111. Here is what we got:

$$\begin{aligned} 111 &= 3 \cdot 37 \\ 1111 &= 7 \cdot 11 \cdot 13 \cdot 101 \\ 111111 &= 3 \cdot 7 \cdot 11 \cdot 13 \cdot 37 \end{aligned}$$

We discussed on the way what a prime number is, how to decide if a number is divisible by 3, 9, or 11, how to figure out if a number n is prime, in particular, why it is enough to check if it is divisible by all primes up to $\sqrt{n}$. We will talk about all these things in detail later. I didn't ask you to factor 11111 and 1111111 here, the reason being that the prime factors are large and not so easy to find. This can be done by hand or, for example, using a computer algebra system Maple. I typed $ifactor(11111)$ and got $41 \cdot 271$. For 1111111 the factorization is $239 \cdot 4649$.

Example 2. Let p be a prime number, $p > 3$. Show that $24 \mid (p-1)(p+1)$. The vertical line here means "divides". Since p is a prime greater than 2, it has to be odd, hence both $(p+1)$ and $(p-1)$ are even. Notice that they are consecutive evens, hence one of them is divisible by 4 and their product is divisible by 8. It remains to show that $3 \mid (p+1)(p-1)$. We have three consecutive numbers $p-1, p, p+1$ hence one of them is divisible by 3. Since $p > 3$ is prime, it cannot be divisible by 3 and hence either $3 \mid (p-1)$ or $3 \mid (p+1)$ and $3 \mid (p-1)(p+1)$. We get $24 \mid (p-1)(p+1)$.

Question: If $6 \mid a$ and $4 \mid a$, does this imply $24 \mid a$? The answer is no (let a=12). Further question — Why did this work with 3 and 8?

Given two integers a and b, we can divide one by the other with a remainder:

$$a = qb + r$$

the quotient q indicates how many times b fits into a and the remainder r is what's left after that, and hence $0 \leq r < b$.

Example 3. Find all integers x such that after you divide x by 7 the remainder is equal to the quotient.

Example 4. When m is divided by 13 and by 15 we get same quotients. In the first case the remainder is 8 while in the second it's zero. Find m.