Fundamental Concepts of Algebra
Spring 2008

Notes 2/5/08
Proving Inequalities

Before we were finding the values of the variable that satisfy a given inequality, now we will be trying to establish the inequality with possibly some given restrictions on the variable(s).

Example 1. Show

$$\frac{x^2+y^2}{2} \geq xy$$

for every value of x and y.

Here is a chain of equivalent inequalities.

$$\begin{aligned} \frac{x^2+y^2}{2} &\geq xy \\ x^2+y^2 &\geq 2xy \\ x^2-2xy+y^2 &\geq 0 \\ (x-y)^2 &\geq 0 \end{aligned}$$

The last inequality in this chain obviously holds for all values of x and y. Note that the inequality turns into equality if and only if $x = y$.

Example 2. Show that the geometric mean of two positive numbers x and y does not exceed their arithmetic mean

$$\frac{x+y}{2} \geq \sqrt{xy}$$

As above, we obtain a chain of equivalent inequalities

$$\begin{aligned} \frac{x+y}{2} &\geq \sqrt{xy} \\ x+y &\geq 2\sqrt{xy} \\ x-2\sqrt{x}\sqrt{y}+y &\geq 0 \\ (\sqrt{x}-\sqrt{y})^2 &\geq 0 \end{aligned}$$

where last inequality obviously holds for all positive x and y, turning into equality if and only if $x = y$.

Example 3. Show that the arithmetic mean of two numbers a ad b does exceed their *quadratic mean*:

$$\frac{a+b}{2} \leq \sqrt{\frac{a^2+b^2}{2}}$$

To get rid of the radical it would be nice to square both sides, but we have to be careful here. For example, squaring a true number inequality $-3 < 2$ we obtain $9 < 4$! This wouldn't happen if both sides of the original inequality were positive.

Let's look at the inequality for the means. The right-hand side is always positive, while the left-hand side could be negative, but if it is negative, the inequality is obvious - we have something negative on the left which is obviously less than something positive on the right. Now, lets assume that both sides are non-negative and square the inequality:

$$\begin{aligned} \frac{a+b}{2} &\leq \sqrt{\frac{a^2+b^2}{2}} \\ \frac{a^2+2ab+b^2}{4} &\leq \frac{a^2+b^2}{2} \\ a^2+2ab+b^2 &\leq 2a^2+2b^2 \\ a^2-2ab+b^2 &\geq 0 \\ (a-b)^2 &\geq 0 \end{aligned}$$

Notice that combining two last examples, for positive a and b we get an inequality for all three means:

$$\sqrt{ab} \leq \frac{a+b}{2} \leq \sqrt{\frac{a^2+b^2}{2}}$$

which turns into equality if and only if $a = b$.

Example 4. Show that for positive x

$$x + \frac{1}{x} \geq 2$$

Example 5. Prove for positive a, b, c, d

$$\sqrt{(a+c)(b+d)} \geq \sqrt{ab} + \sqrt{cd}$$

Example 6. Show for all x and y that

$$x^2 + y^2 + 1 \geq xy + x + y.$$

We will multiply the inequality by 2 and take all the terms to left-hand side

$$\begin{aligned} 2x^2+2y^2+2-2xy-2x-2y &\geq 0 \\ (x^2-2xy+y^2)+(x^2-2x+1)+(y^2-2y+1) &\geq 0 \\ (x-y)^2+(x-1)^2+(y-1)^2 &\geq 0 \end{aligned}$$

The inequality turns into equality if and only if $x = y = 1$.

Example 7. Given that $(a+b+c) = 0$ show that $ab+bc+ac \leq 0$.

We square $(a+b+c) = 0$ and open the parentheses:

$$0 = (a+b+c)^2 = a^2+b^2+c^2+2ab+2bc+2ac$$

Hence

$$ab+bc+ac = \frac{1}{2}\left(-a^2-b^2-c^2\right) \leq 0$$

Inequalities Involving Absolute Value

Recall the definition of the absolute value:

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

Absolute value of x is the distance from x to 0. The graph of $|x|$ is a V-shape with the vertex at the origin. To obtain the graph of $|f(x)|$ you need to draw the graph of $f(x)$ and then flip the parts that are below x-axis.

Example 8. Draw the graph of $f(x) = |x^2 - 3x - 4|$.

Example 9. Solve the inequality

$$|x + 3| < 5$$

This inequality is equivalent to

$$\begin{aligned} -5 < \quad & x + 3 & < 5 \\ -8 < \quad & x & < 2 \end{aligned}$$

Answer: $(-8, 2)$