Algebra for High School Teachers
Fall 2008

Homework 1
Due Wednesday, Sept 3

Problem 1. Use mathematical induction to show that for all integers $n \geq 1$

$$1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

Problem 2. Use mathematical induction to show that for all integers $n \geq 1$ and all real numbers $r \neq 1$

$$1 + r + r^2 + \cdots + r^n = \frac{1 - r^{n+1}}{1 - r}$$

Problem 3. A line divides a plane into two regions. Two crossing lines divide the plane into four regions. Three lines divide the plane into seven regions if there are no parallel lines and no three lines pass through the same point. Let P_n be the number of regions into which n lines in *general* position (no parallel lines and no three lines pass through the same point) divide the plane. Find an explicit formula for P_n.

Problem 4. Fix an integer k. Use mathematical induction to show that for all integers $n \geq k$

$$\binom{k}{k} + \binom{k+1}{k} + \binom{k+2}{k} + \cdots + \binom{n}{k} = \binom{n+1}{k+1}$$

Hint: Here the basis step is $n = k$. For the induction step you will need the Pascal's triangle property (the Lemma that we proved in class).

Here is what this formula means in terms of the Pascal's triangle. Start, say, at the forth 1 on the right side of the triangle then move diagonally down to 4, then farther dow to 10, to 20, 35. Sum up all these numbers. Now move from 35 diagonally to the right. The number there is 70, which is exactly the sum that we found!

Problem 5. Can the sum of three natural numbers be divisible by each of these three numbers? Hint: The answer is yes.

Problem 6. Write four 1s, three 2s, and three 3s around the circle so that the sum of any three numbers in a row is not divisible by 3.

Here is a template that I want you to follow when you write an induction argument.
Show that for all integers $n \geq 1$

$$1^3 + 2^3 + \cdots + n^3 = \left[\frac{n(n+1)}{2}\right]^2$$

Basis step Check this formula for $n = 1$: Indeed, $1^3 = (1 \cdot 2/2)^2$.
Inductive step Suppose that the formula holds true for $n = k$:

$$1^3 + 2^3 + \cdots + k^3 = \left[\frac{k(k+1)}{2}\right]^2, \tag{0.1}$$

and show that the formula holds for $n = k + 1$:

$$1^3 + 2^3 + \cdots + k^3 + (k+1)^3 = \left[\frac{(k+1)(k+2)}{2}\right]^2$$

Add $(k+1)^3$ to the both sides of (**??**). Get

$$\begin{aligned}
1^3+2^3+\cdots+k^3+(k+1)^3 &= \left[\frac{k(k+1)}{2}\right]^2+(k+1)^3 \\
&= (k+1)^2\left[\frac{k^2}{4}+k+1\right] = (k+1)^2\frac{k^2+4k+4}{4} \\
&= \frac{(k+1)^2(k+2)^2}{4} = \left[\frac{(k+1)(k+2)}{2}\right]^2,
\end{aligned}$$

which proves the desired formula.