Algebra for High School Teachers
Fall 2008

Homework 6
Due Wednesday, Oct 8

Explain all your work. Correct answer with no work shown may receive zero credit.

Problem 1. Let A and B be 2×2 matrices with real entries. Show that

$$\det(AB) = (\det A)(\det B).$$

Problem 2. Find two matrices $A,\ B \in M_{2\times 2}(\mathbb{R})$ such that $AB = 0$ while $BA \neq 0$.

Problem 3.

Definition: For integer $n \geq 1$, $\varphi(n)$ denotes the number of positive integers less than or equal to n that are relatively prime to n. This function $\varphi(n)$ is called the *Euler φ-function.*
(a) Find $\varphi(19)$ and $\varphi(5^3)$.
(b) Find $\varphi(p^k)$ where p is a prime and k is a positive integer.

Problem 4.
(a) Show that $2^{15} - 1$ is composite.
(b) Show that $a^{mn} - 1$ is composite. Here a, m, n are integers greater than or equal to 2.

Problem 5. The set

$$\mathbb{Q}[\sqrt{5}] = \left\{a + b\sqrt{5} \mid a, b \in \mathbb{Q}\right\}$$

is a commutative ring with unity with respect to usual addition and multiplication (you don't need to check this). Show that $\mathbb{Q}[\sqrt{5}]$ is a field. For this, you have to show that the multiplicative inverse of every nonzero $a + b\sqrt{5} \in \mathbb{Q}[\sqrt{5}]$ is also of the form $c + d\sqrt{5}$ for some rational c and d.

Problem 6. Let $\mathbb{R}^2$ be the set of all pairs of real numbers

$$\mathbb{R}^2 = \{(a, b) \mid a, b \in \mathbb{R}\}.$$

The coordinatewise addition and multiplication are defined by

$$\begin{aligned}(a_1, b_1) + (a_2, b_2) &= (a_1 + a_2, b_1 + b_2)\\ (a_1, b_1)(a_2, b_2) &= (a_1a_2, b_1b_2)\end{aligned}$$

The set $\mathbb{R}^2$ equipped with these operations is a commutative ring with unity. What is the zero element? What is the additive inverse for (a, b)? Explain. What is the unity element? Explain. Describe the units. Explain.

Problem 7. Let $\mathbb{R}^2$ be the set of all pairs of real numbers

$$\mathbb{R}^2 = \{(a, b) \mid a, b \in \mathbb{R}\}.$$

The addition is still coordinatewise as in the problem above but multiplication is defined by

$$(a_1, b_1) * (a_2, b_2) = (a_1a_2 - b_1b_2, a_1b_2 + a_2b_1)$$

The set $\mathbb{R}^2$ equipped with these operations is a commutative ring. (You don't need to show this.) What is the unity element? Explain. What is the multiplicative inverse for $(a, b) \neq (0, 0)$? Explain. Find (x, y) such that $(x, y) * (x, y) = (-1, 0)$. Explain.

Problem 8. In a substitution code, we replace each letter of the alphabet with another letter. Here is a message intercepted from the communication between King Arthur and King Bela.

U GXUAY LS ZXMEKW AMG TGGTIY HMD TAMGXSD LSSY, FEG GXSA LUGX HEKK HMDIS. FSKT

Can you decrypt it?