Algebra for High School Teachers
Fall 2008

Homework 7 = Exam Practice
Due Wednesday, Oct 15

Exam covers: GCD, prime decomposition, divisibility tests, modular arithmetic, $\mathbb{Z}_m$, Chinese Remainder Theorem, rings and fields, RSA encryption (general picture), complex numbers. Explain all your work. Correct answer with no work shown may receive zero credit.

Problem 1. Find a non-zero 2×2 matrix A such that $A^2 = 0$.

Problem 2. Show that $5 \mid (2^{n+1} + 3^{3n+1})$ for any integer $n \geq 0$.

Problem 3.
(a) Compute the gcd of 1234 and 123 using the Euclidean Algorithm and express the gcd as a linear combination of 1234 and 123.
(b) Show that $\gcd(3n, 3n+2) = 1$ for any odd integer n.

Problem 4. Find the remainder of 93^{25} after division by 7. Do not try to evaluate 93^{25}.

Problem 5. Show that $1729 \mid a^{37} - a$ for any integer a.

Problem 6. Is the ring
$$\mathbb{Z}_5[i] = \{a + bi \,:\, a, b \in \mathbb{Z}_5, i^2 = -1\}$$
an integral domain? Explain.

Problem 7. Let z, w be complex numbers.
(a) Prove that $|zw| = |z||w|$.
(b) Prove that $z\overline{z} = |z|^2$, where $\overline{z}$ is the conjugate of z.

Problem 8. Compute $(1+i)^{99}$.

Problem 9. Solve the equation
$$z^2 + \bar{z} = 0$$

Problem 10. Find a two-digit number which is equal to twice the product of its digits. Show that there are no other two-digit numbers that satisfy this condition.

Problem 11. Find the largest four-digit number divisible by 2, 5, 9, and 11 if all its digits are distinct.