SPHERICAL GEOMETRY AND EULER'S FORMULA

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1. Introduction

We will talk about spherical geometry, i.e. geometry on a sphere. It turns out that all the notions we had on the plane (such as points, straight lines, angles, triangles etc.) make sense in spherical geometry. However, some of the facts we use without a thought in plane geometry are no longer true in spherical geometry. For example, the sum of the angles of a triangle on a sphere is not π! In fact, the greater the area of the triangle is the greater the difference between the sum of the angles and π is (Theorem 2.3).

We will also prove Euler's theorem which says that no mater what convex polyhedron you take, if you count the number of its vertices, subtract the number of its edges, and add the number of its faces you will always get 2. (Check it yourself for tetrahedron or cube!) In our proof of Euler's theorem we will use spherical geometry.

2. Sum of angles of a spherical triangle

The main property of a straight line on the plane is that any segment AB of it is the *shortest* path between points A and B. We use this property to define straight lines (people call them *geodesic lines*, or simply *geodesics*[1]) on a sphere. What do geodesics on a sphere look like? It is not hard to prove (although we will omit it) that geodesics are sections of the sphere with planes passing through the center. Such sections are called *great circles.* For any two points there is a unique great circle passing through them, unless they are opposite. If they are opposite (i.e. belong to a diameter) there are infinitely many such great circles.

Definition 2.1. The length of the arc segment of the great circle between A and B (i.e. the length of the geodesic between A and B) is called the *spherical distance* from A to B.

Once we have defined the analogy to straight lines on a sphere, the definition of a triangle (or any polygon) carries over from the plane to a sphere: the vertices are points on a sphere and the sides are arc segments of great circles. We will call triangles (or polygons) on a sphere *spherical triangles (or polygons).*

As we discussed above each geodesic comes from a section with a plane passing through the origin. Let two geodesics AB and BC start at the same point B. The *angle ABC*

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[1] The name "geodesic" comes from the Greek root "geo-" which means "the earth". As the earth looks very much like a sphere we all are part of a spherical geometry.

between the two geodesics is the angle (the one that is no greater than π) between the corresponding half-planes containing the common point B.

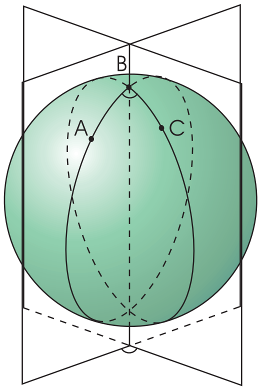

FIGURE 1. The angle between two geodesics

The polygon with smallest number of vertices in the plane is a triangle. On a sphere it is a “two-angle”, i.e., a spherical segment. Indeed, on a sphere if you issue two straight lines (remember that straight lines for us are arcs of great circles) from a point A they will intersect at another point A', the point opposite to A. It’s not hard to see what the area of a spherical segment is.

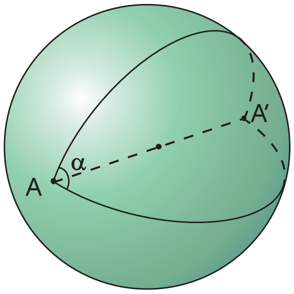

FIGURE 2. Spherical segment

Theorem 2.2. *Let AA' be a spherical segment formed by two geodesics. Let α be the angle between them. Then the area of the segment is equal to $2\alpha R^2$, where R is the radius of the sphere.*

Proof. Clearly, the area of the segment (we denote it S_α) is proportional to the area of the sphere and is proportional to the angle α. So $S_\alpha = c\alpha(4\pi R^2)$. To find coefficient c let us notice that when $\alpha = \pi$ we should get the area of the semi-sphere, i.e. $2\pi R^2$. Therefore, $2\pi R^2 = c\pi(4\pi R^2)$, so $c = 1/2\pi$ and $S_\alpha = 2\alpha R^2$. □

Now we are able to calculate the area of a triangle. Let ABC be a spherical triangle with angles α, β and γ. If we continue the sides of the triangle they will meet at the other three points A', B' and C', opposite to A, B and C, respectively. The triangle $A'B'C'$ is opposite to triangle ABC, so it has the same area as ABC.

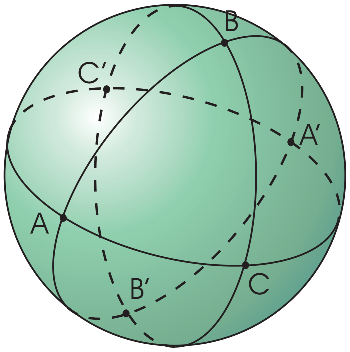

FIGURE 3. The area of a spherical triangle

The rest of the surface of the sphere is covered by three spherical segments: $ABA'C'$ with angle $\pi-\alpha$, $BCB'A'$ with angle $\pi-\beta$, and $CAC'B'$ with angle $\pi-\gamma$. These segments are not overlapping, so we can write:

$$S_{ABC}+S_{A'B'C'}+S_{\pi-\alpha}+S_{\pi-\beta}+S_{\pi-\gamma}=4\pi R^2.$$

Since $S_{ABC}=S_{A'B'C'}$, using Theorem 2.2 we get:

$$2S_{ABC}=4\pi R^2-2(\pi-\alpha)R^2-2(\pi-\beta)R^2-2(\pi-\gamma)R^2.$$

Finally,

$$S_{ABC}=(\alpha+\beta+\gamma-\pi)R^2.$$

We thus proved the following theorem.

Theorem 2.3. *Let ABC be a spherical triangle with angles α, β and γ. Then*

$$\alpha+\beta+\gamma=\pi+\frac{S_{ABC}}{R^2},$$

where S_{ABC} is the area of the triangle and R is the radius of the sphere. In particular, the sum of the angles is greater than π.

Remark 2.4. It is also useful to get a formula for the sum of the external angles of a spherical triangle. Let $\hat{\alpha}=\pi-\alpha$, $\hat{\beta}=\pi-\beta$ and $\hat{\gamma}=\pi-\gamma$ be the external angles of a triangle ABC. Then

$$\hat{\alpha}+\hat{\beta}+\hat{\gamma}=2\pi-\frac{S_{ABC}}{R^2}. \tag{2.1}$$

This follows directly from Theorem 2.3.

3. Sum of external angles of a spherical polygon

Now we are going to generalize (2.1) to any spherical polygon. We will show that for every spherical polygon the sum of its external angles is equal to 2π minus its area divided by R^2.

Theorem 3.1. *Let $P = A_1A_2\dots A_n$ be a spherical polygon. Let $\hat\alpha_1, \dots, \hat\alpha_n$ be its external angles. Then*

$$\hat\alpha_1 + \dots + \hat\alpha_n = 2\pi - \frac{S_P}{R^2},$$

where S_P is the area of P and R is the radius of the sphere.

Proof. The proof is by induction on n. Consider the polygon $P' = A_1A_2\dots A_{n-1}$ obtained from P by removing the last vertex A_n. Then the external angles $\hat\alpha'_i$ of P' are the same as of P except for α'_1 and α'_{n-1}. It easy to see (Figure 4) that

$$\hat\alpha'_1 = \hat\alpha_1 + \angle A_1, \quad \hat\alpha'_i = \hat\alpha_i, \ (2 \le i \le n-2) \quad \hat\alpha'_{n-1} = \hat\alpha_{n-1} + \angle A_{n-1},$$

where $\angle A_1$ and $\angle A_{n-1}$ are the corresponding angles of triangle $\triangle A_1A_{n-1}A_n$.

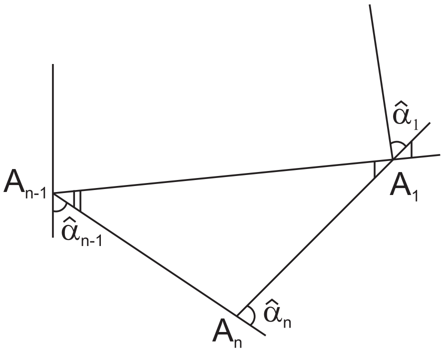

Figure 4. External angles

By inductive assumption

$$\hat\alpha'_1 + \dots + \hat\alpha'_{n-1} = 2\pi - \frac{S_{P'}}{R^2}.$$

Therefore

$$(\hat\alpha_1 + \angle A_1) + \dots + (\hat\alpha_{n-1} + \angle A_{n-1}) + \hat\alpha_n = 2\pi - \frac{S_{P'}}{R^2} + \hat\alpha_n,$$

Since $\hat\alpha_n = \pi - \angle A_n$ we get

$$\hat\alpha_1 + \dots + \hat\alpha_n = 2\pi - \frac{S_{P'}}{R^2} - (\angle A_1 + \angle A_{n-1} + \angle A_n - \pi).$$

By Theorem 2.3, $\angle A_1 + \angle A_{n-1} + \angle A_n = \pi + S_{A_1A_{n-1}A_n}/R^2$, thus we get

$$\hat\alpha_1 + \dots + \hat\alpha_n = 2\pi - \frac{S_{P'}}{R^2} - \frac{S_{A_1A_{n-1}A_n}}{R^2} = 2\pi - \frac{S_P}{R^2}.$$

$\square$

4. Euler's Theorem.

In this section we will give a proof of Euler's theorem based on the results from spherical geometry.

Theorem 4.1. (Euler's Theorem) *Let* Δ *be a convex polyhedron. Let* V, E, *and* F *denote the number of its vertices, edges, and faces respectively. Then*

$$V - E + F = 2.$$

Proof. Let us place the polyhedron Δ inside a sphere centered at point O, where O is some point inside Δ. Project Δ from the center O on the surface of the sphere. You can think of this as if the polyhedron is made of rubber and you pump air into it until it becomes a balloon with vertices and edges drawn on it.

Now the sphere is covered by F spherical polygons $P_1, \ldots, P_F$ that come from the faces of Δ. Thus the sum of areas of these polygons is 4π. (We assume that the radius of the sphere is 1.) Using Theorem 3.1 we get

$$4\pi = S_{P_1} + \cdots + S_{P_F} = 2\pi F - \sum \hat{\alpha}_i, \tag{4.1}$$

where on the right hand side we have the sum of all external angles of all polygons on the sphere.

Let's take a look at this sum from a different point of view. At each vertex v the sum of *internal* angles is 2π:

$$\alpha_1 + \cdots + \alpha_{E_v} = 2\pi,$$

where E_v denotes the number of edges that begin at vertex v. But $\alpha_i = \pi - \hat{\alpha}_i$, thus for each vertex v

$$\hat{\alpha}_1 + \cdots + \hat{\alpha}_{E_v} = \pi E_v - 2\pi.$$

Therefore, the sum of all external angles of all polygons is

$$\sum_v (\pi E_v - 2\pi) = \pi \left(\sum_v E_v \right) - 2\pi V.$$

Since every edge connects exactly two vertices $\sum_v E_v = 2E$.

Finally, from (4.1) we get

$$4\pi = 2\pi F - 2\pi E + 2\pi V, \quad \text{i.e.} \quad V - E + F = 2.$$

□