Advanced Problem Solving II
Jenya Soprunova
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Homework 5
Stable Marriages

Problem 1. Given n boys and n girls, how many ways are there to break them into n pairs?

Problem 2. You wish to assign four girls, A, B, C, and D, to two rooms, two per room. Come up with preference lists for the girls, where each of them ranks the remaining three as possible roommates (no ties allowed), such that no stable assignment exists.

Problem 3. Let A, B be two girls and α, β two boys. Come up with preference lists such that both possible pairings are stable.

Problem 4. Show that Gale-Shapley algorithm for pairing up n boys and n girls takes at most $n^2 - n + 1$ rounds. (Hint: In each round at least one boy goes down his list. What happens in the first round?)

Problem 5. For the following ranking matrix

	A	B	C	D
α	1,4	2,3	3,1	4,2
β	4,1	2,1	3,3	1,3
γ	1,3	2,4	3,4	4,1
δ	4,2	3,2	2,2	1,4

find all the feasible partners for α and then determine $\alpha's$ optimal and pessimal partners. (Do not run the Gale-Shapley algorithm here.)

If α and A are together, A is extremely unhappy, as she is with her last choice. She would run away with anyone interested. Whoever is γ with he would want to run away with A, as she is his first choice. Since in this case A and γ would run away, any such pairing is unstable, so A is not a feasible partner for α.

Let's check that the pairing $\alpha B, \beta D$, γA, δC is stable. It's enough to check that all the boys will stay faithful. Clearly, β and γ will not run away as they are with their first choices. Next, α would only run away with his first choice A, but A would not be interested. Finally, δ would only run away with D, but he is number four on D's list. Hence the marriage αB is feasible.

Similarly, it's easy to check that the pairing $\alpha C, \beta D$, γA, δB is stable, so the marriage αC is feasible.

If α and D are together, α and C would run away, so αD is not feasible.

We have shown that there are two feasible partners for α, namely B and C. Out of the two, B is optimal and C is pessimal.

Problem 6. Assume that among n boys and n girls, there is a boy and a girl who rank each other first. Show that there is only one feasible marriage for each of them, the one where they are together.

Problem 7. Run the Gale-Shapley algorithm for the ranking matrix below twice. First time let the boys ($\alpha, \beta, \gamma, \delta$) propose, and then let the girls (A, B, C, D) propose.

	A	B	C	D
α	1,3	2,3	3,2	4,3
β	1,4	4,1	3,3	2,2
γ	2,2	1,4	3,4	4,1
δ	4,1	2,2	3,1	1,4

Conclude that there is only one stable pairing for this ranking matrix. (Use the theorem that the Gale-Shapley algorithm where boys propose is male-optimal and female-pessimal.)

Problem 8. What is the largest number of rounds in the Gale-Shapley algorithm for 2 boys and 2 girls? (Prove the bound an then show by example that the bound is attainable.)

Problem 9. Come up with a 3×3 ranking matrix for which Gale-Shapley algorithm takes 5 rounds. Explain why this is the largest number of rounds for $n = 3$.

Problem 10. For the following ranking matrix

$$\begin{bmatrix} 1, n & 2, n-1 & 3, n-2 & \dots & n, 1 \\ n, 1 & 1, n & 2, n-1 & \dots & n-1, 2 \\ n-1, 2 & n, 1 & 1, n & \dots & n-2, 3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 2, n-1 & 3, n-2 & 4, n-3 & \dots & 1, n \end{bmatrix}$$

consider the pairing where each girl gets her kth choice for some fixed k with $1 \leq k \leq n$. Show that such a paring is stable. Hint: consider two pairs (A, α) and (B, β) where the husbands are kth choices of their wives and show that A and β would not run away. What are the numbers of the wives on their husbands' lists? It is useful to notice that for each entry i, j in the ranking matrix we have $i + j = n + 1$.