Advanced Problem Solving II
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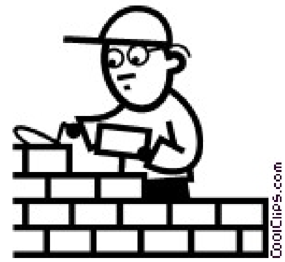

Bricks and Boxes

Here A, B, C, a, b, c are postive integers.

Problem 1. Show that a $5 \times 6 \times 1$ box can be filled with $2 \times 3 \times 1$ bricks but cannot be filled with such bricks that are in parallel position.

Problem 2. Suppose that $b > a$, but b is not a multiple of a. Show that a $(a+b) \times ab \times 1$ box can be filled with $a \times b \times 1$ bricks but cannot be filled with such bricks that are in parallel position.

Problem 3. We say that an $A \times B \times C$ box is a *multiple* of an $a \times b \times c$ brick if there are integers p, q and r such that pa, qb, rc is a rearrangement of A, B, C.

(a) Is the $10 \times 16 \times 17$ box a multiple of the $1 \times 2 \times 4$ brick?
(b) Let $A \times B \times C$ be a multiple of $a \times b \times c$. Show that one can fill $A \times B \times C$ box with $a \times b \times c$ bricks so that all the bricks are in parallel position.

Problem 4. Recall that for a real number θ

$$e^{i\theta} = \cos\theta + i \sin\theta.$$

Use this definition to compute (a) $e^{\pi i}$, (b) $e^{\frac{\pi i}{2}}$, and (c) $e^{2\pi i}$.

Problem 5. Show that $e^{\frac{2A\pi}{a} i} = 1$ exactly when A is a multiple of a.

Problem 6. Let A and $a > 1$ be positive integers. Use formula for the sum of the geometric sequence to compute

$$S(A) := e^{\frac{2\pi}{a} i} + e^{\frac{2\cdot 2\pi}{a} i} + e^{\frac{3\cdot 2\pi}{a} i} + \cdots + e^{\frac{A\cdot 2\pi}{a} i}.$$

Show that $S(A)$ equals zero if and only if A is a multiple of a.

Problem 7. Let N and $a > 1$ be positive integers. Use either the previous problem or sum of geometric sequence formula to show that

$$e^{\frac{2\pi N}{a} i} + e^{\frac{2\pi(N+1)}{a} i} + e^{\frac{2\pi(N+2)}{a} i} + \cdots + e^{\frac{2\pi(N+a-1)}{a} i} = 0.$$

Using the definition for $S(A)$, $S(B)$, and $S(C)$ from problem 6, we see that their product equals

$$S(A)S(B)S(C) = \sum_{k=1}^{A} \sum_{l=1}^{B} \sum_{m=1}^{C} e^{\frac{2\pi i(k+l+m)}{a}},$$

where each triple (k, l, m) corresponds to a $1 \times 1 \times 1$ cubie in the $A \times B \times C$ box. Suppose that the $A \times B \times C$ box is filled with $a \times b \times c$ bricks. Since each of such bricks can be filled with $a \times 1 \times 1$ bricks, the same is true about the initial box. Then in the summation above we can collect terms that correspond to one such brick. By Problem 7 these terms will sum up to zero, which proves that $S(A)S(B)S(C) = 0$.

Problem 8. Given that $S(A)S(B)S(C) = 0$ prove that one of A, B, or C is a multiple of a.

We have proved

Theorem 1. *If the $A \times B \times C$ box can be filled with $a \times b \times c$ bricks then a divides one of A, B, or C and same is true about b and c.*

Note that this does not mean that $A \times B \times C$ is a multiple $a \times b \times c$. For example, the $5 \times 6 \times 1$ cube can be filled with $2 \times 3 \times 1$ bricks, although one is not a multiple of the other.

Problem 9. We say that a brick $a \times b \times c$ is *harmonic* if we can rearrange the numbers a, b, and c so that the first of them divides the second and the second divides the third. Decide whether (a) $6 \times 2 \times 18$ and (b) $12 \times 2 \times 8$ are harmonic.

Problem 10. Assume that a $A \times B \times 1$ box is filled with harmonic $a \times b \times 1$ bricks where a divides b. Show that $A \times B \times 1$ is a multiple of $a \times b \times 1$. (Use the Theorem to show that b divides one of A, B and common sense to explain why a divides both A and B).

Problem 11. Now we will address a 3D version of the previous problem. Assume that an $A \times B \times C$ box is filled with harmonic $a \times b \times c$ bricks where a divides b and b divides c. Prove that $A \times B \times C$ box is a multiple of the $a \times b \times c$ brick. For this, use the Theorem to conclude that c divides one of A, B, C and suppose that c divides C. Next, consider an $A \times B \times 1$ face of the box and explain why it can be filled with $a \times b \times 1$ bricks. The result now follows from the previous problem.

Problem 12. Show that if a brick $a \times b \times c$ with $a \leq b \leq c$ is not harmonic then there is an $A \times B \times C$ box which can be filled with $a \times b \times c$ bricks, without being a multiple of the brick. (Hint: If a does not divide b, try $(a + b) \times ab \times c$ box. What to do if a divides b, but b does not divide c?)